

REGULARITY OF BICYCLIC GRAPHS AND THEIR POWERS

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ABSTRACT. Let $I(G)$ be the edge ideal of a bicyclic graph G with a dumbbell as the base graph. In this paper, we characterize the Castelnuovo-Mumford regularity of $I(G)$ in terms of the induced matching number of G . For the base case of this family of graphs, i.e. dumbbell graphs, we explicitly compute the induced matching number. Moreover, we prove that $\text{reg } I(G)^q = 2q + \text{reg } I(G) - 2$, for all $q \geq 1$, when G is a dumbbell graph with a connecting path having no more than two vertices.

INTRODUCTION

Let I be a homogeneous ideal in a polynomial ring $R = \mathbb{k}[x_1, \dots, x_r]$ over a field \mathbb{k} . The Castelnuovo-Mumford regularity of I , denoted by $\text{reg}(I)$, has been an interesting and active research topic for the past decades. There exists a vast literature on the study of $\text{reg}(I)$ (see e.g. [9]). A celebrated result on the behavior of the regularity of powers of ideals was given independently by Cutkosky, Herzog, and Trung in [10], and by Kodiyalam in [24]. In both papers, it is proved that for all $q \geq q_0$, the regularity of the powers of I is asymptotically a linear function $\text{reg}(I^q) = dq + b$, where q_0 is the so-called stabilizing index, and b is the so-called constant. The value of d in the above formula is well understood (see [28, Theorem 3.2]). For example, d is equal to the degree of the generators of I when I is equigenerated in degree d (see loc. cit.). However, there is no general or precise method to determine q_0 and b .

In recent years, many researchers have tried to compute q_0 and b for special families of ideals. The most simple case, yet interesting, is when I is the edge ideal of a finite simple graph. Let $G = (V(G), E(G))$ denote a finite simple undirected graph. Let R be the polynomial ring $\mathbb{k}[x_i \mid x_i \in V(G)]$ where \mathbb{k} is any field. The edge ideal $I(G)$ of G is the ideal

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E(G)).$$

Several authors have settled the problem of determining the stabilizing index and the constant for special families of graphs. Banerjee proved that $\text{reg } I(G)^q = 2q$, for all $q \geq 2$, when G is a gap-free and cricket-free graph (see [4]). Beyarslan, Hà, and Trung settled the problem for the family of forests and cycles (see [6]). Moghimian, Fakhari, and Yassemi answered the question for the family of whiskered cycles (see [26]). Their results were expanded to the family of unicyclic graphs by Alilooee, Beyarslan, and Selvaraja (see [2]). Moreover, Alilooee and Banerjee determined the stabilizing index and the constant for the family of bipartite graphs with regularity equal to three (see [1]). Jayanthan and Selvaraja settled the problem for the family of very well-covered graphs (see [21]). Recently, Erey proved that if G is a gap-free and diamond-free graph, then $\text{reg } I(G)^q = 2q$ for all $q \geq 2$ (see [13]). The approach is focused on the relations between the combinatorics of graphs

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and algebraic properties of edge ideals. We refer the reader to see [5], [23], [17], [8], [20], [3], [29] and [27] for more information on this topic. The main purpose of this paper is to extend the results of [2] to the family of bicyclic graphs (i.e. a graph with two cycles).

The family of bicyclic graphs has three possible types of base graphs: two cycles joined at a vertex, two cycles connected by a path and two cycles sharing a path. In [15], Gu computed the regularity of all the powers of the edge ideal of a base graph of the first type (i.e. two cycles joined at a vertex).

In this paper, we consider the family of bicyclic graphs where the base graph is a dumbbell. A dumbbell graph $C_n \cdot P_l \cdot C_m$ is a graph consisting of two cycles C_n and C_m connected with a path P_l , where n , m , and l are the number of vertices (see Example 2.1). For convenience of notation, we define the following function

$$\xi_3(n) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod{3}, \\ 0 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Here, we describe the basic outline and main results of this paper.

In Section 1, we fix some notations and recall known results which are important to our approach.

In Section 2, we use combinatorial techniques to compute the induced matching number of a dumbbell graph. Then, applying inductive methods, we study the regularity of the edge ideals of dumbbell graphs. For a dumbbell graph $C_n \cdot P_l \cdot C_m$, we assume that “ $n \bmod 3 \leq m \bmod 3$ ”. Since the graphs $C_n \cdot P_l \cdot C_m$ and $C_m \cdot P_l \cdot C_n$ are clearly isomorphic, the cases “ $n \equiv 2 \pmod{3}$, $m \equiv 0, 1 \pmod{3}$ ” have the same results as the cases “ $n \equiv 0, 1 \pmod{3}$, $m \equiv 2 \pmod{3}$ ”. Our approach is based on the Lozin transformation (see [25] and [7]), and the induced matching number of a dumbbell graph. The following results are given in this section:

Theorem A (Theorem 2.4). *Let $n, m \geq 3$ and $l \geq 1$, then*

$$\nu(C_n \cdot P_l \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) - \xi_3(m) + 1}{3} \right\rfloor.$$

Theorem B (Theorem 2.16). *Let $m, n \geq 3$ and $l \geq 1$,*

(i) *if $l \equiv 0, 1 \pmod{3}$, then*

$$\text{reg } I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & \text{if } n, m \equiv 2 \pmod{3}, \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise;} \end{cases}$$

(ii) *if $l \equiv 2 \pmod{3}$, then*

$$\text{reg } I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & \text{if } n \equiv 0, 1 \pmod{3}, m \equiv 2 \pmod{3} \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

In Section 3, for a bicyclic graph G having a dumbbell graph as the base, we give a combinatorial characterization of $\text{reg } I(G)$ in terms of the induced matching number $\nu(G)$.

Theorem C (Theorem 3.24). *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. The following statements hold.*

(I) *Let $n, m \equiv 0, 1 \pmod{3}$, then $\text{reg } I(G) = \nu(G) + 1$.*

(II) *Let $n \equiv 0, 1 \pmod{3}$ and $m \equiv 2 \pmod{3}$, then*

$$\nu(G) + 1 \leq \text{reg } I(G) \leq \nu(G) + 2,$$

and $\text{reg } I(G) = \nu(G) + 2$ if and only if $\nu(G) = \nu(G \setminus \Gamma_G(C_m))$.

- (III) Let $n, m \equiv 2 \pmod{3}$ and $l \geq 3$, then $\nu(G) + 1 \leq \text{reg } I(G) \leq \nu(G) + 3$. Moreover:
- (i) $\text{reg } I(G) = \nu(G) + 3$ if and only if $\nu(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(G)$.
 - (ii) $\text{reg } I(G) = \nu(G) + 1$ if and only if the following conditions hold:
 - (a) $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$;
 - (b) $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$;
 - (c) $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$.
- (IV) Let $n, m \equiv 2 \pmod{3}$ and $l \leq 2$, then $\nu(G) + 1 \leq \text{reg } I(G) \leq \nu(G) + 2$. If x is a vertex on P_l and $\mathcal{L}_x(G)$ is the Lozin transformation of G with respect to x , then $\text{reg } I(G) = \nu(G) + 1$ if and only if the following conditions are satisfied:
- (a) $\nu(\mathcal{L}_x(G)) - \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n \cup C_m)) > 1$;
 - (b) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n))$;
 - (c) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_m))$.

In [Section 4](#), we investigate the asymptotic behavior of regularity of powers of $I(C_n \cdot P_l \cdot C_m)$ when $l \leq 2$. The approach takes advantage of the notion of even-connectedness and the relations between the induced matching number of graphs and the regularity of the edge ideal.

Theorem D ([Theorem 4.7](#)). *Let $C_n \cdot P_l \cdot C_m$ with $l \leq 2$, then*

$$\text{reg } I(C_n \cdot P_l \cdot C_m)^q = 2q + \text{reg } I(C_n \cdot P_l \cdot C_m) - 2$$

for any $q \geq 1$.

The above equality is no longer true for the case $l \geq 3$ as there are immediate counter examples (see [Remark 4.9](#)).

1. PRELIMINARIES

Let $R = \mathbb{k}[x_1, \dots, x_r]$ be a standard graded polynomial ring over a field \mathbb{k} and let $\mathfrak{m} = (x_1, \dots, x_r)$ be its maximal irrelevant ideal. For a graded R -module M , one can define the Castelnuovo-Mumford regularity in different ways. We recall the definition of the regularity of an R -module M given in terms of the minimal free resolution of M . The *minimal graded free resolution of M* is an exact sequence of the form

$$0 \rightarrow F_p \rightarrow F_{p-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0,$$

where each F_i is a graded free R -module of the form $F_i = \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{i,j}(M)}$, each $\varphi_i : F_i \rightarrow$

F_{i-1} , with $F_{-1} := M$, is a graded homomorphism of degree zero such that $\varphi_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i$ for all $i \geq 0$. The number $\beta_{i,j}(M)$, called the $(i, j)^{\text{th}}$ -graded Betti number of M , is an important invariant of the module M . In particular, the number $\beta_i(M) = \sum_{j \in \mathbb{N}} \beta_{i,j}(M)$ is

called the *i -th Betti number of M* . Note that the minimal free resolution of M is unique up to isomorphism, hence the graded Betti numbers are uniquely determined.

Definition 1.1. *Let M be a finitely generated graded R -module. The regularity of M is given by*

$$\text{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

Remark 1.2. *Note that, if I is a graded ideal of R , then $\text{reg}(R/I) = \text{reg}(I) - 1$.*

Let $G = (V, E)$ be a graph with vertex set $V = \{x_1, \dots, x_r\}$. Here, we recall some classes of graphs that we need for this study.

Definition 1.3. *Let $G = (V, E)$ be a graph.*

- (i) G is called a *path on l vertices*, denoted by P_l , if $V = \{x_1, \dots, x_l\}$ and $\{x_i, x_{i+1}\} \in E$ for all $1 \leq i \leq l-1$.
- (ii) G is called a *cycle on n vertices*, denoted by C_n , if $V = \{x_1, \dots, x_n\}$ and $\{x_i, x_{i+1}\} \in E$ for all $1 \leq i \leq n-1$ and $\{x_n, x_1\} \in E$.
- (iii) G is called a *dumbbell graph* if G contains two cycles C_n and C_m joined by a path P_l on l vertices. We denote it by $C_n \cdot P_l \cdot C_m$ (see [Example 2.1](#)).

For a vertex u in a graph $G = (V, E)$, let $N_G(u) = \{v \in V \mid \{u, v\} \in E\}$ be the set of *neighbors* of u , and set $N_G[u] := N_G(u) \cup \{u\}$. An edge e is *incident* to a vertex u if $u \in e$. The degree of a vertex $u \in V$, denoted by $\deg_G(u)$, is the number of edges incident to u . When there is no confusion, we omit G and write $N(u)$, $N[u]$ and $\deg(u)$. For an edge e in a graph $G = (V, E)$, we define $G \setminus e$ to be the subgraph of G obtained by deleting e from E (but the vertices are maintained).

Let $G = (V, E)$ be a graph and $W \subseteq V$, the *induced subgraph of G on W* , denoted by $G[W]$, is the graph with vertex set W and edge set $\{e \in E \mid e \subseteq W\}$. For a subset $W \subseteq V$ of the vertices in G , we define $G \setminus W$ to be the induced subgraph of G obtained by deleting the vertices of W and their incident edges from G . When $W = \{u\}$ consists of a single vertex, we write $G \setminus u$ instead of $G \setminus \{u\}$. For an edge $e = \{u, v\} \in E$, let $N_G[e] = N_G[u] \cup N_G[v]$ and define G_e to be the induced subgraph of G over the vertex set $V \setminus N_G[e]$.

One can think of the vertices of $G = (V, E)$ as the variables of the polynomial ring $R = \mathbb{k}[x_1, \dots, x_r]$ for convenience. Similarly, the edges of G can be considered as square free monomials of degree two. By an abuse of notation, we use e to refer to both the edge $e = \{x_i, x_j\} \in E$ and the monomial $e = x_i x_j \in I(G)$.

Definition 1.4. Let $G = (V, E)$ be a graph.

A collection C of edges of G is called a *matching* if the edges in C are pairwise disjoint. The maximum size of a matching in G is called its *matching number*, which is denoted by $\text{match}(G)$.

A collection C of edges of G is called an *induced matching* if C is a matching, and C consists of all edges of the induced subgraph $G[\bigcup_{e \in C} e]$ of G . The maximum size of an induced matching in G is called its *induced matching number* and it is denoted by $\nu(G)$.

Remark 1.5. [6, Remark 2.12] Let P_l be a path on l vertices, then we have

$$\nu(P_l) = \left\lfloor \frac{l+1}{3} \right\rfloor$$

Remark 1.6. [6, Remark 2.13] Let C_n be a cycle on n vertices, then we have

$$\nu(C_n) = \left\lfloor \frac{n}{3} \right\rfloor.$$

Depending on $r = n \bmod 3$ we can assume the following:

- (i) when $r = 0$, there exists a maximal induced matching of C_n that does not contain the edges $x_1 x_2$ and $x_1 x_n$;
- (ii) when $r = 1$, there exists a maximal induced matching of C_n that does not contain the edges $x_1 x_2$, $x_1 x_n$ and $x_{n-1} x_n$;
- (iii) when $r = 2$, there exists a maximal induced matching of C_n that does not contain the edges $x_1 x_2$, $x_2 x_3$, $x_1 x_n$ and $x_{n-1} x_n$.

Theorem 1.7. [16, Lemma 3.1, Theorems 3.4 and 3.5] Let $G = (V, E)$ be a graph.

- (i) If H is an induced subgraph of G , then $\text{reg } I(H) \leq \text{reg } I(G)$;

(ii) Let $x \in V$, then

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G \setminus x), \operatorname{reg} I(G \setminus N[x]) + 1\};$$

(iii) Let $e \in E$, then

$$\operatorname{reg} I(G) \leq \max\{2, \operatorname{reg} I(G \setminus e), \operatorname{reg} I(G_e) + 1\}.$$

Now we recall the concept of even-connection introduced by Banerjee in [4].

Definition 1.8 ([4]). Let $G = (V, E)$ be a graph with edge ideal $I = I(G)$. Two vertices x_i and x_j in G are called even-connected with respect to an s -fold product $M = e_1 \cdots e_s$, where e_1, \dots, e_s are edges in G , if there is a path p_0, \dots, p_{2l+1} , for some $l \geq 1$, in G such that the following conditions hold:

- (i) $p_0 = x_i$ and $p_{2l+1} = x_j$;
- (ii) for all $0 \leq j \leq l-1$, $\{p_{2j+1}, p_{2j+2}\} = e_i$ for some i ;
- (iii) for all i , $|\{j \mid \{p_{2j+1}, p_{2j+2}\} = e_i\}| \leq |\{t \mid e_t = e_i\}|$.

Theorem 1.9. [4, Theorems 6.1 and 6.5] Let $M = e_1 e_2 \cdots e_s$ be a minimal generator of I^s . Then $(I^{s+1} : M)$ is minimally generated by monomials of degree 2, and uv (u and v may be the same) is a minimal generator of $(I^{s+1} : M)$ if and only if either $\{u, v\} \in E$ or u and v are even-connected with respect to M .

Remark 1.10. [4, Lemma 6.11] Let $(I^{s+1} : M)^{\text{pol}}$ be the polarization of the ideal $(I^{s+1} : M)$ (see e.g. [19, §1.6]). From the previous theorem we can construct a graph G' whose edge ideal is given by $(I^{s+1} : M)^{\text{pol}}$. The new graph G' is given by:

- (i) All the vertices and edges of G .
- (ii) Any two vertices u, v , $u \neq v$ that are even-connected with respect to M are connected by an edge in G' .
- (iii) For every vertex u which is even-connected to itself with respect to M , there is a new vertex u' which is connected to u by an edge and not connected to any other vertex (so uu' is a whisker).

Theorem 1.11. [4, Theorem 5.2] Let G be a graph and $\{m_1, \dots, m_r\}$ be the set of minimal monomial generators of $I(G)^q$ for all $q \geq 1$, then

$$\operatorname{reg} I(G)^{q+1} \leq \max\{\operatorname{reg} (I(G)^q : m_l) + 2q, 1 \leq l \leq r, \operatorname{reg} I(G)^q\}.$$

We recall a result by Kalai and Meshulam on the regularity of monomial ideals.

Theorem 1.12. ([22], [18]) Let I_1, \dots, I_s be monomial ideals in R , then

$$\operatorname{reg} \left(R / \sum_{i=1}^s I_i \right) \leq \sum_{i=1}^s \operatorname{reg} (R / I_i).$$

In the particular case of edge ideals we have the following upper bound.

Corollary 1.13. Let G be a simple graph. If G_1, \dots, G_s are subgraphs of G such that $E(G) = \bigcup_{i=1}^s E(G_i)$ then

$$\operatorname{reg} (R / I(G)) \leq \sum_{i=1}^s \operatorname{reg} (R / I(G_i)).$$

The previous upper bound is sharp when G is a disjoint union of the graphs G_1, \dots, G_s .

Corollary 1.14. [5, Corollary 3.10] *Let G be a simple graph. If G can be written as a disjoint union of graphs G_1, \dots, G_s then*

$$\operatorname{reg}(R/I(G)) = \sum_{i=1}^s \operatorname{reg}(R/I(G_i)).$$

The regularity of the edge ideal of a forest was first computed by Zheng in [30, Theorem 2.18].

Theorem 1.15. [30, Theorem 2.18] *Let G be a forest, then*

$$\operatorname{reg} I(G) = \nu(G) + 1.$$

In [23] Katzman proved that the previous equality is a lower bound for any graph.

Theorem 1.16. [23, Corollary 1.2] *Let G be a graph, then*

$$\operatorname{reg} I(G) \geq \nu(G) + 1.$$

The decycling number of a graph is an important combinatorial invariant which can be used to obtain an upper bound for the regularity of the edge ideal of a graph.

Definition 1.17. *For a graph G and $D \subset V(G)$, if $G \setminus D$ is acyclic, i.e. contains no induced cycle, then D is said to be a decycling set of G . The size of a smallest decycling set of G is called the decycling number of G and denoted by $\nabla(G)$.*

Theorem 1.18. [7, Theorem 4.11] *Let G be a graph, then*

$$\operatorname{reg} I(G) \leq \nu(G) + \nabla(G) + 1.$$

In [6] Beyarslan, Hà and Trung provided a formula for the regularity of the powers of edge ideals of forests and cycles in terms of the induced matching number.

Theorem 1.19. [6, Theorem 4.7] *Let G be a forest, then*

$$\operatorname{reg} I(G)^q = 2q + \nu(G) - 1.$$

for all $q \geq 1$.

Theorem 1.20. [6, Theorem 5.2]. *Let C_n be a cycle with n vertices, then*

$$\operatorname{reg} I(C_n) = \begin{cases} \nu(C_n) + 1 & \text{if } n \equiv 0, 1 \pmod{3}, \\ \nu(C_n) + 2 & \text{if } n \equiv 2 \pmod{3}, \end{cases}$$

where $\nu(C_n) = \lfloor \frac{n}{3} \rfloor$ denote the induced matching number of C_n . Moreover,

$$\operatorname{reg} I(C_n)^q = 2q + \nu(C_n) - 1.$$

and for all $q \geq 2$.

In addition, the authors of [6] gave a lower bound for the regularity of the powers of the edge ideal of an arbitrary graph, and an upper bound for the regularity of the edge ideal of a graph containing a Hamiltonian path.

Theorem 1.21. [6, Theorem 4.5] *Let G be a graph and let $\nu(G)$ denote its induced matching number. Then, for all $q \geq 1$, we have*

$$\operatorname{reg} I(G)^q \geq 2q + \nu(G) - 1.$$

Theorem 1.22. [6, Theorem 3.1] *Let G be a graph on n vertices. If G contains a Hamiltonian path, then*

$$\operatorname{reg} I(G) \leq \left\lfloor \frac{n+1}{3} \right\rfloor + 1.$$

2. REGULARITY AND INDUCED MATCHING NUMBER OF A DUMBBELL GRAPH

In this section we compute the induced matching number of a dumbbell graph and the regularity of its edge ideal. Recall that $C_n \cdot P_l \cdot C_m$ denotes the graph constructed by joining two cycles C_n and C_m via a path P_l . In this section, we denote the vertices of C_n , C_m and P_l by $\{x_1, \dots, x_n\}$, $\{y_1, \dots, y_m\}$ and $\{z_1, \dots, z_l\}$, respectively. We make the identifications $x_1 = z_1$ and $y_1 = z_l$.

Example 2.1. Two base cases when $l = 2$ and $l = 1$ are the following:

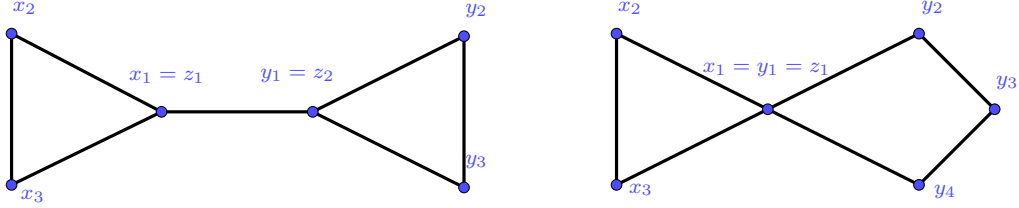
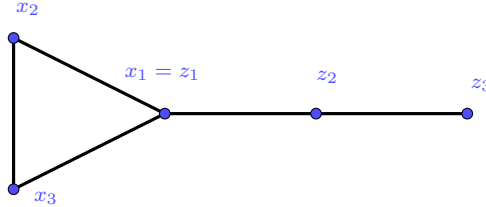


FIGURE 1. The graphs $C_3 \cdot P_2 \cdot C_3$ and $C_3 \cdot P_1 \cdot C_4$.

Notation 2.2. Let ξ_3 be the function defined as below

$$\xi_3(n) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod{3}, \\ 0 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Let $C_n \cdot P_l$ be the graph given by connecting the path P_l to the cycle C_n . For instance, the graph $C_3 \cdot P_3$ can be illustrated as the following:



Proposition 2.3. Let $n \geq 3$ and $l \geq 1$, then

$$\nu(C_n \cdot P_l) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) + 1}{3} \right\rfloor.$$

Proof. Case 1: From [Remark 1.6](#), in the case $n \equiv 2 \pmod{3}$ we have that in clockwise and anticlockwise directions the two consecutive edges to the vertex x_1 are not chosen in a maximal induced matching of C_n . Then, we can choose the edges in P_l without any constraint coming from the maximal induced matching chosen in C_n , and so we have $\nu(C_n \cdot P_l) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{l+1}{3} \right\rfloor$.

Case 2: It remains to consider the case $\xi_3(n) = 1$, i.e., $n \equiv 0, 1 \pmod{3}$. Let \mathcal{M} be an induced matching of maximal size in G . We analyze separately the two cases of whether $z_1 z_2$ (the edge adjacent to the cycle C_n) is in \mathcal{M} or not.

Suppose $z_1 z_2$ is not an edge of \mathcal{M} . Then \mathcal{M} can be considered as the union of a maximal matching of C_n as introduced in [Remark 1.6](#) and a maximal matching of the path $P_l \setminus z_1$. Thus $|\mathcal{M}| = \nu(C_n) + \nu(P_{l-1}) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{(l-1)+1}{3} \right\rfloor$.

If $z_1 z_2 \in \mathcal{M}$, then none of the edges incident to the vertices in $N_{C_n}[x_1] = \{x_1, x_2, x_n\}$ are in $\mathcal{M}|_{C_n} := \{e \in \mathcal{M} \mid e \subset C_n\}$. Hence $|\mathcal{M}|_{C_n} = \nu(P_{n-3})$, and since $n \equiv 0, 1 \pmod{3}$

then it follows $|\mathcal{M}|_{C_n} = \lfloor \frac{n-2}{3} \rfloor = \lfloor \frac{n}{3} \rfloor - 1$. From $z_1 z_2 \in \mathcal{M}$ we get $|\mathcal{M}|_{P_l} = \nu(P_l) = \lfloor \frac{l+1}{3} \rfloor$. So, by joining both computations we get $|\mathcal{M}| = \lfloor \frac{n}{3} \rfloor - 1 + \lfloor \frac{l+1}{3} \rfloor = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{l-2}{3} \rfloor$.

Therefore, we obtain that $\nu(C_n \cdot P_l) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{(l-1)+1}{3} \rfloor$. \square

Theorem 2.4. *Let $n, m \geq 3$ and $l \geq 1$, then*

$$\nu(C_n \cdot P_l \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) - \xi_3(m) + 1}{3} \right\rfloor.$$

Proof. We use the same argument as in [Proposition 2.3](#). By [Remark 1.6](#) we have that when either $n \equiv 2 \pmod{3}$ or $m \equiv 2 \pmod{3}$, then the maximal induced matching in C_n or in C_m does not affect the way we choose edges in the path P_l .

In the case $n \equiv 0, 1 \pmod{3}$, we can choose a maximal induced matching that does not use the edge $z_1 z_2$ by [Remark 1.6](#), i.e., the extreme vertex z_1 on the path P_l does not appear in the induced matching. Similarly, when $m \equiv 0, 1 \pmod{3}$ we can drop the other extreme vertex. \square

The aim of the rest of this section is to explicitly compute the regularity of $I(C_n \cdot P_l \cdot C_m)$ in terms of the induced matching number. We divide it into three subsections depending on the value of $l \pmod{3}$. The base of our computations is given by the following proposition.

Proposition 2.5. *Let $n, m \geq 3$ and $l \geq 1$, then*

$$\text{reg } I(C_n \cdot P_l \cdot C_m) - \nu(C_n \cdot P_l \cdot C_m) = \text{reg } I(C_n \cdot P_{l+3} \cdot C_m) - \nu(C_n \cdot P_{l+3} \cdot C_m).$$

Proof. From the formula obtained in [Theorem 2.4](#) or [\[25, Lemma 1\]](#), we have the equality

$$\nu(C_n \cdot P_{l+3} \cdot C_m) = \nu(C_n \cdot P_l \cdot C_m) + 1.$$

We can apply the Lozin transformation (see e.g. [\[25\]](#), [\[7\]](#)) to any of the vertices in the bridge P_l , then from [\[7, Theorem 1.1\]](#) we have

$$\text{reg } I(C_n \cdot P_{l+3} \cdot C_m) = \text{reg } I(C_n \cdot P_l \cdot C_m) + 1.$$

Thus, the statement of the proposition follows by subtracting these equalities. \square

From the previous proposition, it follows that we only need to consider the cases $l = 1$, $l = 2$ and $l = 3$. We treat each case in a separate subsection.

The basic approach in the next three subsections is to obtain lower and upper bounds that coincide.

2.1. The case $l = 1$. Throughout this subsection, we consider the dumbbell graph $C_n \cdot P_1 \cdot C_m$.

Proposition 2.6. *Let $n, m \geq 3$, then*

$$\text{reg } I(C_n \cdot P_1 \cdot C_m) \leq \max \left\{ \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1, \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 2 \right\}.$$

Moreover, $\text{reg } I(C_n \cdot P_1 \cdot C_m)$ is equal to one of these terms.

Proof. We use [\[11, Lemma 3.2\]](#) that gives an improved version of the exact sequence that comes from deleting the vertex z_1 and its neighbors. We have

$$\text{reg } I(C_n \cdot P_1 \cdot C_m) \in \left\{ \text{reg } I((C_n \cdot P_1 \cdot C_m) \setminus z_1), \text{reg } I((C_n \cdot P_1 \cdot C_m) \setminus N[z_1]) + 1 \right\}.$$

Since $(C_n \cdot P_1 \cdot C_m) \setminus z_1 = P_{n-1} \cup P_{m-1}$ and $(C_n \cdot P_1 \cdot C_m) \setminus N[z_1] = P_{n-3} \cup P_{m-3}$, we get the result by applying [Theorem 1.15](#). \square

Theorem 2.7. *Let $n, m \geq 3$, then*

$$\text{reg } I(C_n \cdot P_1 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_1 \cdot C_m) + 2 & \text{if } n \equiv 2 \pmod{3}, m \equiv 2 \pmod{3}; \\ \nu(C_n \cdot P_1 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. Suppose $n \equiv 2 \pmod{3}$ and $m \equiv 2 \pmod{3}$. Since $\lfloor \frac{k-2}{3} \rfloor = \lfloor \frac{k}{3} \rfloor$ when $k \equiv 2 \pmod{3}$, we have

$$\max\{\lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1, \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{m-2}{3} \rfloor + 2\} = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2.$$

Thus [Proposition 2.6](#) yields

$$(1) \quad \text{reg } I(C_n \cdot P_1 \cdot C_m) \leq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2.$$

Consider the induced subgraph $H = (C_n \cdot P_1 \cdot C_m) \setminus \{x_n\}$ where x_n is in C_n and it is incident to x_1 (e.g. see x_3 in [Example 2.1](#)). In fact, H is the graph given by joining C_m and a path P_{n-1} , that is, $H = C_m \cdot P_{n-1}$. Now from [Proposition 2.3](#), we have that $\nu(H) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$. By [Theorem 1.7\(i\)](#), we get $\text{reg } I(C_n \cdot P_1 \cdot C_m) \geq \text{reg } I(H)$. From [\[2, Theorem 1.2\]](#), we have $\text{reg } I(H) = \nu(H) + 2$. Therefore, the equality holds in (1). The proof of this part is complete since [Theorem 2.4](#) yields $\nu(C_n \cdot P_1 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$.

For any case distinct to $n \equiv 2 \pmod{3}$ and $m \equiv 2 \pmod{3}$, we have

$$\max\{\lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1, \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{m-2}{3} \rfloor + 2\} = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1.$$

Therefore, from [Proposition 2.6](#), we have

$$(2) \quad \text{reg } I(C_n \cdot P_1 \cdot C_m) \leq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1.$$

From [Theorem 2.4](#), we have $\nu(C_n \cdot P_1 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$. Moreover, [Theorem 1.16](#) gives $\text{reg } I(C_n \cdot P_1 \cdot C_m) \geq \nu(C_n \cdot P_1 \cdot C_m) + 1$. Thus, the equality in (2) holds. Therefore the proof is complete. \square

2.2. The case $l = 2$. Throughout this subsection, we consider the dumbbell graph $C_n \cdot P_2 \cdot C_m$.

Remark 2.8. *The regularity of $I(C_n)$ is given in [Theorem 1.20](#). For simplicity of notation, we use the equivalent formula $\text{reg } I(C_n) = \lfloor \frac{n-2}{3} \rfloor + 2$. Similarly, we have $\text{reg } (R/I(C_n)) = \lfloor \frac{n-2}{3} \rfloor + 1$.*

Proposition 2.9. *Let $n, m \geq 3$, then*

$$(3) \quad \nu(C_n \cdot P_2 \cdot C_m) \leq \text{reg} \left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)} \right) \leq \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 2.$$

Proof. We only need to prove the inequality on the right since the lower bound is given due to [Theorem 1.16](#). In the original graph $C_n \cdot P_2 \cdot C_m$ we remove the edge that connects the two cycles C_n and C_m . The set of vertices of C_n and C_m are given respectively by $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$, and we assume that the edge $e = x_1 y_1$ is the bridge between the two cycles. We denote by $C_n \cup C_m$ the resulting graph given as the disjoint union of the two cycles C_n and C_m .

Note that $(C_n \cdot P_2 \cdot C_m) \setminus e = C_n \cup C_m$ and that $I((C_n \cdot P_2 \cdot C_m)_e) = I(P_{n-3} \cup P_{m-3})$ because $N_G[x_1] \cup N_G[y_1] = \{x_1, x_2, x_n, y_1, y_2, y_m\}$, where P_{n-3} is the path on the vertices

$\{x_3, \dots, x_{n-1}\}$ and P_{m-3} is the path on the vertices $\{y_3, \dots, y_{m-1}\}$. Thus, [Theorem 1.7\(iii\)](#) gives the inequality

$$\operatorname{reg} \left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)} \right) \leq \max \left\{ \operatorname{reg} \left(\frac{R}{I(C_n \cup C_m)} \right), \operatorname{reg} \left(\frac{R}{I(P_{n-3} \cup P_{m-3})} \right) + 1 \right\}.$$

From [Corollary 1.14](#) and [Remark 2.8](#), it follows that

$$\operatorname{reg} \left(\frac{R}{I(C_n \cup C_m)} \right) = \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 2.$$

Similarly, [Theorem 1.15](#) and [Remark 1.5](#) give that

$$\operatorname{reg} \left(\frac{R}{I(P_{n-3} \cup P_{m-3})} \right) + 1 = \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 1.$$

This proves the proposition. \square

As a result of the previous proposition, we can prove the following corollary.

Corollary 2.10. *If $n \equiv 0, 1 \pmod{3}$ and $m \equiv 0, 1 \pmod{3}$, then*

$$\operatorname{reg} \left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)} \right) = \nu(C_n \cdot P_2 \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor$$

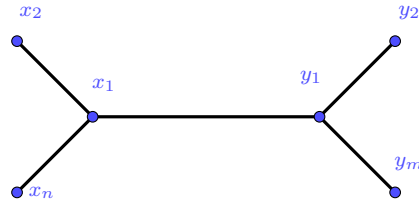
Proof. Note that $\left\lfloor \frac{k}{3} \right\rfloor = \left\lfloor \frac{k-2}{3} \right\rfloor + 1$ when $k \equiv 0, 1 \pmod{3}$. From [Theorem 2.4](#), in [\(3\)](#) the lower and upper bound coincide for these cases. So the equality is established. \square

Now we have only three more cases left to deal with, i.e., the case $n \equiv 0 \pmod{3}$, $m \equiv 2 \pmod{3}$, the case $n \equiv 1 \pmod{3}$, $m \equiv 2 \pmod{3}$, and the case $n \equiv 2 \pmod{3}$, $m \equiv 2 \pmod{3}$.

Lemma 2.11. *If $n \equiv 2 \pmod{3}$ and $m \equiv 2 \pmod{3}$, then*

$$\operatorname{reg} \left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)} \right) = \nu(C_n \cdot P_2 \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Proof. We shall divide the graph into three subgraphs H_1 , H_2 and H_3 . We make $H_1 = C_n \setminus \{x_1\}$ and $H_2 = C_m \setminus \{y_1\}$. The subgraph H_3 is defined by taking the bridge $e = x_1y_1$ and the neighboring vertices $\{x_2, x_n, y_2, y_m\}$, i.e. the graph below.



Using this decomposition and [Theorem 1.12](#) we get the inequality

$$\operatorname{reg} R/I(C_n \cdot P_2 \cdot C_m) \leq \operatorname{reg} (R/I(H_1)) + \operatorname{reg} (R/I(H_2)) + \operatorname{reg} (R/I(H_3)),$$

where H_1 and H_2 are paths of length $n-1$ and $m-1$, respectively, and using [Theorem 1.15](#) we get

$$\operatorname{reg} R/I(C_n \cdot P_2 \cdot C_m) \leq \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Finally, in the present case $n \equiv 2 \pmod{3}$ and $m \equiv 2 \pmod{3}$ we have the equality $\nu(C_n \cdot P_2 \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1$, and so the proof follows from [Theorem 1.16](#). \square

Lemma 2.12. *If $n \equiv 0, 1 \pmod{3}$ and $m \equiv 2 \pmod{3}$, then*

$$\text{reg} \left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)} \right) = \nu(C_n \cdot P_2 \cdot C_m) + 1 = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Proof. In this case we delete the vertex x_1 from the cycle C_n . We have that $H = (C_n \cdot P_2 \cdot C_m) \setminus \{x_1\}$ is an induced subgraph of $C_n \cdot P_2 \cdot C_m$ which is given as the disjoint union of a path of length $n - 1$ and C_m , i.e. $H = P_{n-1} \cup C_m$. From [Theorem 1.7\(i\)](#), [Corollary 1.14](#), [Theorem 1.15](#) and [Theorem 1.20](#) we get that

$$\text{reg}(R/I(C_n \cdot P_2 \cdot C_m)) \geq \text{reg}(R/I(H)) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

It follows from [Proposition 2.9](#) and the fact that $\lfloor k/3 \rfloor = \lfloor (k-2)/3 \rfloor + 1$ when $k \equiv 0, 1 \pmod{3}$ that

$$\text{reg} R/I(C_n \cdot P_2 \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

So we are through. \square

Theorem 2.13. *Let $n, m \geq 3$, then*

$$\text{reg} I(C_n \cdot P_2 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_2 \cdot C_m) + 2 & \text{if } n \equiv 0, 1 \pmod{3}, m \equiv 2 \pmod{3}; \\ \nu(C_n \cdot P_2 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. It follows from [Corollary 2.10](#), [Lemma 2.11](#) and [Lemma 2.12](#). \square

2.3. The case $l = 3$. Throughout this subsection, we consider the dumbbell graph $C_n \cdot P_3 \cdot C_m$.

Proposition 2.14. *Let $n, m \geq 3$, then*

- (i) $\text{reg} I(C_n \cdot P_3 \cdot C_m) \leq \nu(C_n \cdot P_3 \cdot C_m) + 2$, if $n, m \equiv 2 \pmod{3}$;
- (ii) $\text{reg} I(C_n \cdot P_3 \cdot C_m) = \nu(C_n \cdot P_3 \cdot C_m) + 1$, otherwise.

Proof. Let $E(P_3) = \{e, e'\}$ be the set of edges of P_3 , where $e = z_1 z_2$ and $e' = z_2 z_3$ are connected to C_n and C_m , respectively. Note that $I((C_n \cdot P_3 \cdot C_m) \setminus e) = I(C_n \cup (e' \cdot C_m))$ and that $I((C_n \cdot P_3 \cdot C_m)_e) = I(P_{n-3} \cup P_{m-1})$ because

$$N_G[e] = \{x_1 = z_1, x_2, x_n, z_2, y_1 = z_3\},$$

where $e' \cdot C_m$ is the unicyclic graph with C_m and a whisker e' attached to C_m , P_{n-3} is the path on the vertices $\{x_3, \dots, x_{n-1}\}$ and P_{m-1} is the path on the vertices $\{y_2, \dots, y_m\}$. Thus, [Theorem 1.7\(iii\)](#) gives the inequality

$$\text{reg} \left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)} \right) \leq \max \left\{ \text{reg} \left(\frac{R}{I(C_n \cup (e' \cdot C_m))} \right), \text{reg} \left(\frac{R}{I(P_{n-3} \cup P_{m-1})} \right) + 1 \right\}.$$

From [Proposition 2.3](#) and [\[2, Lemma 3.2\]](#) follows that $\text{reg}(I(e' \cdot C_m)) = \lfloor \frac{m}{3} \rfloor + \lfloor \frac{3-\xi_3(m)}{3} \rfloor + 1$. Thus, using [Remark 2.8](#), [Corollary 1.14](#) and [Theorem 1.15](#), we get $\text{reg} \left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)} \right) \leq \max \left\{ \left\lfloor \frac{n-2}{3} \right\rfloor + 1 + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{3-\xi_3(m)}{3} \right\rfloor, \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1 \right\}$.

On the other hand, from [Theorem 2.4](#) we have that $\nu(C_n \cdot P_3 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + \lfloor \frac{4-\xi_3(n)-\xi_3(m)}{3} \rfloor$. Therefore, we see that $\text{reg} \left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)} \right) \leq \nu(C_n \cdot P_3 \cdot C_m) + 1$ when $n, m \equiv 2 \pmod{3}$, and that $\text{reg} \left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)} \right) = \nu(C_n \cdot P_3 \cdot C_m)$ in all the remaining cases. \square

Theorem 2.15. *Let $n, m \geq 3$, then*

$$\text{reg } I(C_n \cdot P_3 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_3 \cdot C_m) + 2 & \text{if } n, m \equiv 2 \pmod{3}, \\ \nu(C_n \cdot P_3 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. From [Proposition 2.14](#), it suffices to show that $\text{reg } I(C_n \cdot P_3 \cdot C_m) \geq \nu(C_n \cdot P_3 \cdot C_m) + 2$ when $n, m \equiv 2 \pmod{3}$. Hence, we assume $n, m \equiv 2 \pmod{3}$. Let z_2 be the middle vertex of $C_n \cdot P_3 \cdot C_m$. By deleting z_2 we see that $H = (C_n \cdot P_3 \cdot C_m) \setminus z_2 = C_n \cup C_m$ is an induced subgraph of $C_n \cdot P_3 \cdot C_m$. From [Theorem 1.20](#) and [Corollary 1.14](#), we have that

$$\text{reg } I(H) = \text{reg } I(C_n) + \text{reg } I(C_m) - 1 = \nu(C_n) + \nu(C_m) + 3.$$

Since $\nu(C_n \cdot P_3 \cdot C_m) = \nu(C_n) + \nu(C_m) + 1$, then using [Theorem 1.7\(i\)](#) we get

$$\text{reg } I(C_n \cdot P_3 \cdot C_m) \geq \text{reg } I(H) = \nu(C_n \cdot P_3 \cdot C_m) + 2. \quad \square$$

2.4. Regularity of a dumbbell graph. Now we are ready for the main result of this section. In the following theorem we compute the regularity of the edge ideal of the dumbbell $C_n \cdot P_l \cdot C_m$.

Theorem 2.16. *Let $m, n \geq 3$ and $l \geq 1$, then*

(i) *if $l \equiv 0, 1 \pmod{3}$, then*

$$\text{reg } I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & \text{if } n, m \equiv 2 \pmod{3}, \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise;} \end{cases}$$

(ii) *if $l \equiv 2 \pmod{3}$, then*

$$\text{reg } I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & n \equiv 0, 1 \pmod{3}, m \equiv 2 \pmod{3}; \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. Follows from [Proposition 2.5](#), and [Theorem 2.7](#), [Theorem 2.13](#), and [Theorem 2.15](#). \square

3. COMBINATORIAL CHARACTERIZATION OF $\text{reg } (I(G))$ IN TERMS OF $\nu(G)$

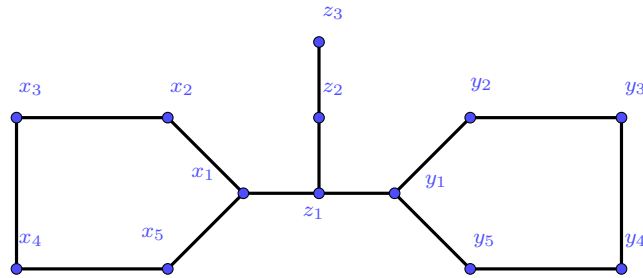
In this section, we focus on any bicyclic graph which admits the dumbbell graph $C_n \cdot P_l \cdot C_m$ as its base bicycle and study the regularity of the edge ideals of these graphs.

Let G be any bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. Then its decycling number is smaller or equal than 2. Thus by [Theorem 1.16](#) and [Theorem 1.18](#), we get

$$\nu(G) + 1 \leq \text{reg } I(G) \leq \nu(G) + 3.$$

There are examples of bicyclic graphs with a dumbbell where the regularity of its edge ideal is equal to $\nu(G) + 1$, $\nu(G) + 2$ and $\nu(G) + 3$.

Example 3.1. *The following graph G*



has $\text{reg } I(G) = 6$ and $\nu(G) = 3$.

In this section, we give a combinatorial characterization of the bicyclic graphs with regularity $\nu(G) + 1$, $\nu(G) + 2$ and $\nu(G) + 3$.

For the rest of the paper, we shall use the term “dumbbell” of the bicyclic graph G , and it denotes the unique subgraph of G of the form $C_n \cdot P_l \cdot C_m$.

The following simple remark will be crucial in our treatment.

Remark 3.2. [2, Observation 2.1] *Let G be a graph with a leaf y and its unique neighbor x , say $e = \{x, y\}$. If $\{e_1, \dots, e_s\}$ is an induced matching in $G \setminus N[x]$, then $\{e_1, \dots, e_s, e\}$ is an induced matching in G . So we have $\nu(G \setminus N[x]) + 1 \leq \nu(G)$.*

Proposition 3.3. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. The following statements hold.*

- (i) *When $n, m \equiv 0, 1 \pmod{3}$, we have $\text{reg } I(G) = \nu(G) + 1$.*
- (ii) *When $n \equiv 0, 1 \pmod{3}$ and $m \equiv 2 \pmod{3}$, we have $\text{reg } I(G) \leq \nu(G) + 2$.*
- (iii) *When $l \leq 2$, we have $\text{reg } I(G) \leq \nu(G) + 2$.*

Proof. (i) Again, it is enough to prove the upper bound $\text{reg } I(G) \leq \nu(G) + 1$. Let E' be the set of edges $E' = E(G) \setminus E(C_n \cdot P_l \cdot C_m)$. We proceed by induction on the cardinality of E' . If $|E'| = 0$ then the statement follows from Theorem 2.16, so we assume $|E'| > 0$. There exists a leaf y in G such that $N[y] = \{x\}$. Let $G' = G \setminus x$ and $G'' = G \setminus N[x]$, then by Theorem 1.7 we have

$$\text{reg } I(G) \leq \max\{\text{reg } I(G'), \text{reg } I(G'') + 1\}.$$

The graphs G' and G'' can be either bicyclic graphs with the same dumbbell $C_n \cdot P_l \cdot C_m$, or the disjoint union of two unicyclic graphs with cycles C_n and C_m , or unicyclic graphs with a cycle C_r ($r = n$ or $r = m$) of the type $r \equiv 0, 1 \pmod{3}$, or forests. Using either the induction hypothesis, or [2, Theorem 1.2] and Corollary 1.14, or [2, Theorem 1.2], or Theorem 1.15, then we get $\text{reg } I(G') = \nu(G') + 1$ and $\text{reg } I(G'') = \nu(G'') + 1$. Since we have $\nu(G') \leq \nu(G)$ and $\nu(G'') + 1 \leq \nu(G)$ (by Remark 3.2), then we obtain the required inequality.

(ii) and (iii) follow by the same inductive argument, only changing the fact that G' and G'' could be unicyclic graphs with cycle C_r of the type $r \equiv 2 \pmod{3}$. \square

Remark 3.4. *The inductive process of the previous proposition cannot conclude $\text{reg } I(G) \leq \nu(G) + 2$ in the case $l \geq 3$. Here we may encounter two disjoint induced subgraphs G_1 and G_2 with $\text{reg } I(G_i) = \nu(G_i) + 2$, which implies $\text{reg } I(G_1 \cup G_2) = \nu(G_1 \cup G_2) + 3$. This is exactly the case of Example 3.1.*

An alternative proof of the inequality $\text{reg } I(G) \leq \nu(G) + 3$ for $l \geq 3$ can be given by using the same inductive technique of Proposition 3.3.

For the rest of the paper we use the following notation.

Notation 3.5. *Let G be a graph, $H \subset G$ be a subgraph, and v and w be vertices of G . Then, we assume the following:*

- (i) $d(v, w)$ denotes the length (i.e., the number of edges) of a minimal path between v and w . In particular, $d(v, v) = 0$.
- (ii) $d(v, H)$ denotes the minimal distance from the vertex v to the subgraph H , that is

$$d(v, H) = \min\{d(v, w) \mid w \in H\}.$$

In particular, $d(v, H) = 0$ if and only if $v \in H$.

- (iii) Let $H' \subset G$ be a subgraph, then the distance between H and H' is given by

$$d(H, H') = \min\{d(v, H') \mid v \in H\}.$$

In particular, $d(H, H') = 0$ if and only if $H \cap H' \neq \emptyset$.

(iv) $\Gamma_G(H)$ denotes the subset of vertices

$$\Gamma_G(H) = \{v \in G \mid d(v, H) = 1\}.$$

(v) In the case $k > 0$, $S_{G,k}(H)$ denotes the induced subgraph given by restricting to the vertex set

$$V(S_{G,k}(H)) = \{v \in G \mid d(v, H) \geq k\}.$$

(vi) $S_{G,0}(H)$ denotes the subgraph given by the vertex set

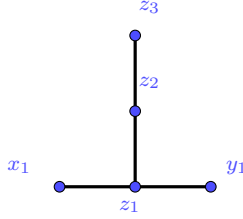
$$V(S_{G,0}(H)) = \{v \in G \mid d(v, H) > 0 \text{ or } \deg(v) \geq 3\}.$$

and the edge set

$$E(S_{G,0}(H)) = \{(v, w) \in E(G) \mid v, w \in V(S_{G,0}(H))\} \\ \setminus \{(v, w) \in E(G) \mid v, w \in H\}.$$

We clarify the previous notation in the following example.

Example 3.6. (i) Let G be the graph of [Example 3.1](#) and $H = C_5 \cup C_5$ be the subgraph given by the two cycles of length 5. Then, we have that $\Gamma_G(H)$ is the set containing the vertex in the middle of the bridge joining the two cycles, that $S_{G,0}(H)$ is a graph of the form

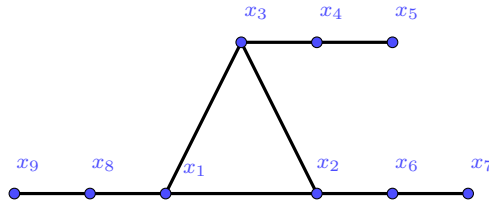


and that the graph

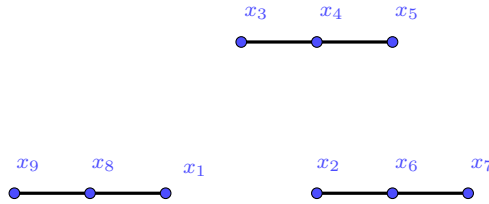


represents $S_{G,2}(H)$.

(ii) Let G be the graph given by



and H be the triangle induced by the vertices $\{x_1, x_2, x_3\}$. Then, we have that $\Gamma_G(H) = \{x_4, x_6, x_8\}$, that $S_{G,0}(H)$ is a graph of the form



and that the graph



represents $S_{G,2}(H)$.

We have already computed $\text{reg } I(G)$ in the case $n, m \equiv 0, 1 \pmod{3}$, for the remaining cases we divide this section into subsections.

3.1. Case I. In this subsection we focus on the case $n \equiv 0, 1 \pmod{3}$ and $m \equiv 2 \pmod{3}$. This case turns out to be almost identical to a unicyclic graph, and our treatment is influenced by [2, Section 3].

Notation 3.7. Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n \equiv 0, 1 \pmod{3}$ and $m \equiv 2 \pmod{3}$. We denote by F_1, \dots, F_c the connected components of $S_{G,0}(C_m)$, and in this case each F_i is either a tree or a unicyclic graph with cycle C_n (and $n \equiv 0, 1 \pmod{3}$). Then, the graph $S_{G,2}(C_m)$ can be given as the union of the components H_1, \dots, H_c , where each one is defined as

$$H_i = F_i \setminus \{v \in G \mid d(v, C_m) \leq 1\}.$$

Note that each H_i can be a disconnected graph or even the empty graph.

Remark 3.8. The following statements hold.

(i) The graph $G \setminus \Gamma_G(C_m)$ has a decomposition of the form

$$G \setminus \Gamma_G(C_m) = C_m \cup \left(\bigcup_{i=1}^c H_i \right),$$

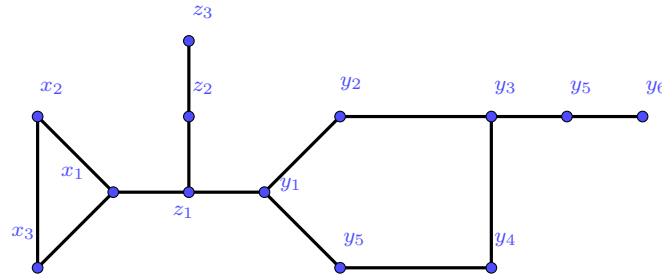
and in particular

$$\nu(G \setminus \Gamma_G(C_m)) = \nu(C_m) + \sum_{i=1}^c \nu(H_i)$$

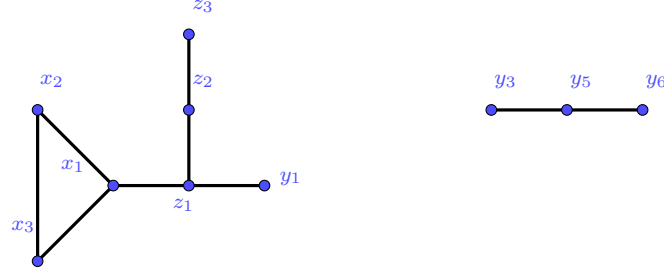
because $d(C_m, H_i) \geq 2$ for all $1 \leq i \leq c$ and $d(H_i, H_j) \geq 2$ for all $1 \leq i < j \leq c$.

(ii) For each $i = 1, \dots, c$, we have that $|F_i \cap C_m| = 1$.

Example 3.9. Let G be the graph



and C_5 be the cycle given by $\{y_1, y_2, y_3, y_4, y_5\}$. We have that $\Gamma_G(C_5) = \{z_1, y_5\}$. The graph $S_{G,0}(C_5)$ is given by



with connected components F_1 (graph on the left) and F_2 (graph on the right). The graph $S_{G,2}(C_5)$ is given by



with connected components H_1 (graph on the left) and H_2 (graph on the right).

Lemma 3.10. Adopt [Notation 3.7](#). If $\nu(H_i) = \nu(F_i)$ for all $1 \leq i \leq c$, then $\nu(G \setminus \Gamma_G(C_m)) = \nu(G)$.

Proof. Follows identically to [\[2, Lemma 3.5\]](#). \square

Proposition 3.11. Adopt [Notation 3.7](#). If $\nu(G \setminus \Gamma_G(C_m)) < \nu(G)$ then $\text{reg } I(G) = \nu(G) + 1$.

Proof. Once more, we shall only prove that $\text{reg } I(G) \leq \nu(G) + 1$. Assume that $\nu(G \setminus \Gamma_G(C_m)) < \nu(G)$, then the contrapositive of [Lemma 3.10](#) implies that there exists some i with $\nu(H_i) < \nu(F_i)$.

Fix i such that $\nu(H_i) < \nu(F_i)$. From [Remark 3.8\(ii\)](#), let x be the vertex in $F_i \cap C_m$. Let us use the notations $G' = G \setminus x$ and $G'' = G \setminus N[x]$. Again, we have the inequality

$$\text{reg } I(G) \leq \max\{\text{reg } I(G'), \text{reg } I(G'') + 1\}.$$

Note that both G' and G'' can be either unicyclic graphs with cycle C_n (and $n \equiv 0, 1 \pmod{3}$), or forests. Hence, from [\[2, Theorem 1.2\]](#) and [Theorem 1.15](#) we get that $\text{reg } I(G') = \nu(G') + 1$ and $\text{reg } I(G'') = \nu(G'') + 1$.

In the case of G' , we have that $\text{reg } I(G') = \nu(G') + 1 \leq \nu(G) + 1$. Let H be the induced subgraph of G obtained by deleting the vertices of $F_i \cup N_G[x]$. Then we have $G'' = H \cup H_i$. Let \mathcal{M}_1 and \mathcal{M}_2 be maximal induced matchings in H and H_i , respectively, then $\nu(G'') = |\mathcal{M}_1| + |\mathcal{M}_2|$ because $d(H, H_i) \geq 2$. By the condition $\nu(F_i) > \nu(H_i)$ then there exists a maximal induced matching \mathcal{M}_3 in F_i , such that $|\mathcal{M}_3| > |\mathcal{M}_2|$. From the fact that $H \cup F_i$ is an induced subgraph in G and $d(H, F_i) \geq 2$, then we get

$$\nu(G) \geq \nu(H \cup F_i) = |\mathcal{M}_1| + |\mathcal{M}_3| > |\mathcal{M}_1| + |\mathcal{M}_2| = \nu(G'').$$

Hence $\text{reg } I(G'') = \nu(G'') + 1 \leq \nu(G)$, and so we get the statement of the proposition. \square

Theorem 3.12. Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n \equiv 0, 1 \pmod{3}$ and $m \equiv 2 \pmod{3}$. Then the following statements hold.

- (i) $\nu(G) + 1 \leq \text{reg } I(G) \leq \nu(G) + 2$;

(ii) $\text{reg } I(G) = \nu(G) + 2$ if and only if $\nu(G) = \nu(G \setminus \Gamma_G(C_m))$.

Proof. In Proposition 3.3 we proved (i). In order to prove (ii), we only need to show that $\nu(G \setminus \Gamma_G(C_m)) = \nu(G)$ implies $\text{reg } I(G) \geq \nu(G) + 2$, because the inverse implication follows from Proposition 3.11.

From Remark 3.8(i), $G \setminus \Gamma_G(C_m) = C_m \cup (\cup_{i=1}^c H_i)$ where each H_i is either a forest or a unicyclic graph with cycle C_n (and $n \equiv 0, 1 \pmod{3}$). Then, from Corollary 1.14, [2, Theorem 1.2] and Theorem 1.15 we get

$$\begin{aligned} \text{reg } I(G \setminus \Gamma_G(C_m)) &= \text{reg } I(C_m) + \text{reg } I(\cup_{i=1}^c H_i) - 1 \\ &= (\nu(C_m) + 2) + (\nu(\cup_{i=1}^c H_i) + 1) - 1 \\ &= \nu(G \setminus \Gamma_G(C_m)) + 2 \\ &= \nu(G) + 2. \end{aligned}$$

Finally, since $G \setminus \Gamma_G(C_m)$ is an induced subgraph of G then we have $\text{reg } I(G) \geq \nu(G) + 2$. \square

3.2. Case II. The object of study of this subsection is the case where $n, m \equiv 2 \pmod{3}$, $l \geq 3$, and in particular when $\text{reg } I(G) = \nu(G) + 3$. More specifically, we shall give necessary and sufficient conditions for the equality $\text{reg } I(G) = \nu(G) + 3$.

Notation 3.13. Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod{3}$ and $l \geq 3$. As in Notation 3.7, let F_1, \dots, F_c be the components of the graph $S_{G,0}(C_n)$. We order the F_i 's in such a way that F_1 is a unicyclic graph with cycle C_m , and for all $i > 1$ we have that F_i is a tree. The graph $S_{G,2}(C_n)$ can be decomposed in components H_1, \dots, H_c where

$$H_i = F_i \setminus \{v \in G \mid d(v, C_n) \leq 1\}.$$

Remark 3.14. From the previous notation get the following simple remarks.

(i) The graph $G \setminus \Gamma_G(C_n)$ has a decomposition of the form

$$G \setminus \Gamma_G(C_n) = C_n \cup \left(\bigcup_{i=1}^c H_i \right),$$

and in particular

$$\nu(G \setminus \Gamma_G(C_n)) = \nu(C_n) + \sum_{i=1}^c \nu(H_i)$$

because $d(C_n, H_i) \geq 2$ for all $1 \leq i \leq c$ and $d(H_i, H_j) \geq 2$ for all $1 \leq i < j \leq c$.

(ii) Similarly, the graph $G \setminus \Gamma_G(C_n \cup C_m)$ has a decomposition of the form

$$G \setminus \Gamma_G(C_n \cup C_m) = C_n \cup \left(\bigcup_{i=2}^c H_i \right) \cup (H_1 \setminus \Gamma_{H_1}(C_m)),$$

and in particular

$$\nu(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(C_n) + \sum_{i=2}^c \nu(H_i) + \nu(H_1 \setminus \Gamma_{H_1}(C_m)).$$

(iii) For each $i = 1, \dots, c$, we have that $|F_i \cap C_n| = 1$.

(iv) The statement of Lemma 3.10 also holds in this case, that is, if $\nu(H_i) = \nu(F_i)$ for all $1 \leq i \leq c$, then $\nu(G \setminus \Gamma_G(C_n)) = \nu(G)$.

(v) Due to the assumption $l \geq 3$, then we have that C_m must be an induced subgraph of H_1 . During this subsection and the next one we shall fundamentally use this fact, and it will allow us to inductively “separate” the two cycles C_n and C_m .

Lemma 3.15. *Adopt Notation 3.13. If $\nu(H_i) = \nu(F_i)$ for all $1 \leq i \leq c$ and $\nu(H_1) = \nu(H_1 \setminus \Gamma_{H_1}(C_m))$, then*

$$\nu(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(G).$$

Proof. Since $G \setminus \Gamma_G(C_n \cup C_m)$ is an induced subgraph of G , then we have $\nu(G \setminus \Gamma_G(C_n \cup C_m)) \leq \nu(G)$. From Remark 3.14(ii) we get

$$\begin{aligned} \nu(G \setminus \Gamma_G(C_n \cup C_m)) &= \nu(C_n) + \sum_{i=2}^c \nu(H_i) + \nu(H_1 \setminus \Gamma_{H_1}(C_m)) \\ &= \nu(C_n) + \sum_{i=2}^c \nu(H_i) + \nu(H_1) \\ &= \nu(C_n) + \sum_{i=1}^c \nu(F_i) \\ &\geq \nu(G), \end{aligned}$$

and so $\nu(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(G)$. \square

Proposition 3.16. *Adopt Notation 3.13. If $\nu(G \setminus \Gamma_G(C_n \cup C_m)) < \nu(G)$, then*

$$\text{reg } I(G) \leq \nu(G) + 2.$$

Proof. From the contrapositive of Lemma 3.15, it follows that there exists some i with $\nu(H_i) < \nu(F_i)$ or we have $\nu(H_1 \setminus \Gamma_{H_1}(C_m)) < \nu(H_1)$. Then we divide the proof into two cases.

Case 1. In this case we assume that for some $1 \leq i \leq c$ we have $\nu(H_i) < \nu(F_i)$. This case follows similarly to Proposition 3.11. Let x be the vertex in $F_i \cap C_n$, let us use the notations $G' = G \setminus x$ and $G'' = G \setminus N[x]$. Once more, we have the inequality

$$\text{reg } I(G) \leq \max\{\text{reg } I(G'), \text{reg } I(G'') + 1\}.$$

Note that both G' and G'' are unicyclic graphs, and so we have $\text{reg } I(G') \leq \nu(G') + 2$ and $\text{reg } I(G'') \leq \nu(G'') + 2$ (see Theorem 1.18). Since we have $\nu(G') \leq \nu(G)$ and $\nu(G'') + 1 \leq \nu(G)$ (see the proof of Proposition 3.11), then the inequality follows in this case.

Case 2. Now we suppose that $\nu(H_1 \setminus \Gamma_{H_1}(C_m)) < \nu(H_1)$. Let x be the vertex in $F_1 \cap C_n$, let us use the notations $G' = G \setminus x$ and $G'' = G \setminus N[x]$. We use the inequality

$$\text{reg } I(G) \leq \max\{\text{reg } I(G'), \text{reg } I(G'') + 1\}.$$

The graphs G' and G'' are unicyclic. For the graph G' we have $\text{reg } I(G') \leq \nu(G') + 2 \leq \nu(G) + 2$.

The graph G'' can be given as the disjoint union of H_1 and another graph H defined by $H = G \setminus (F_1 \cup N[x])$, that is $G'' = H \cup H_1$ and $d(H, H_1) \geq 2$. Thus it follows that $\nu(G'') = \nu(H) + \nu(H_1)$ and that $\text{reg } I(G'') = \text{reg } I(H) + \text{reg } I(H_1) - 1$ (see Corollary 1.14).

Since H is a forest, Theorem 1.15 gives $\text{reg } I(H) = \nu(H) + 1$. From [2, Corollary 3.11], it follows that $\text{reg } I(H_1) = \nu(H_1) + 1$. By summing up, we obtain that $\text{reg } I(G'') \leq \nu(G'') + 1$. So we get the inequality $\text{reg } I(G'') + 1 \leq \nu(G'') + 2 \leq \nu(G) + 2$, because G'' is an induced subgraph of G . \square

Now we are ready to completely describe the case where $\text{reg } I(G) = \nu(G) + 3$.

Theorem 3.17. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. Then $\text{reg } I(G) = \nu(G) + 3$ if and only if the following conditions are satisfied:*

- (i) $n \equiv 2 \pmod{3}$;
- (ii) $m \equiv 2 \pmod{3}$;

- (iii) $l \geq 3$;
- (iv) $\nu(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(G)$.

Proof. In [Proposition 3.3](#) we proved that the conditions (i), (ii) and (iii) are necessary, and from [Proposition 3.16](#) we have that the condition (iv) is also necessary. Hence, we only need to prove that $\text{reg } I(G) = \nu(G) + 3$ under these conditions.

Let $W = G \setminus \Gamma_G(C_n \cup C_m)$. From [Remark 3.14\(ii\)](#) and [Corollary 1.14](#), obtain the equality

$$\text{reg } (I(W)) = \text{reg } (I(C_n)) + \text{reg } (I(\cup_{i=2}^c H_i)) + \text{reg } (I(H_1 \setminus \Gamma_{H_1}(C_m))) - 2.$$

Note that the graph $H_1 \setminus \Gamma_{H_1}(C_m)$ can be given as the disjoint union of the cycle C_m and the forest $H = (H_1 \setminus \Gamma_{H_1}(C_m)) \setminus C_m$, such that $d(H, C_m) \geq 2$. From [Theorem 1.20](#) and [Theorem 1.15](#) we get $\text{reg } I(C_m) = \nu(C_m) + 2$ and $\text{reg } I(H) = \nu(H) + 1$, respectively, and so [Corollary 1.14](#) implies that $\text{reg } I(H_1 \setminus \Gamma_{H_1}(C_m)) = \text{reg } I(C_m) + \text{reg } I(H) - 1 = \nu(C_m) + \nu(H) + 2 = \nu(H_1 \setminus \Gamma_{H_1}(C_m)) + 2$.

Therefore, by also using [Theorem 1.20](#) and [Theorem 1.15](#), we obtain

$$\begin{aligned} \text{reg } (I(W)) &= (\nu(C_n) + 2) + (\nu(\cup_{i=2}^c H_i) + 1) + (\nu(H_1 \setminus \Gamma_{H_1}(C_m)) + 2) - 2 \\ &= \nu(W) + 3 \\ &= \nu(G) + 3. \end{aligned}$$

Since W is an induced subgraph of G then we get

$$\text{reg } I(G) \geq \text{reg } I(W) = \nu(G) + 3,$$

and so from [Theorem 1.18](#) the equality is obtained. \square

3.3. Case III. In this subsection we assume that G is a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod{3}$ and $l \geq 3$. Now that we have characterized when $\text{reg } I(G) = \nu(G) + 3$, then we want to distinguish between $\text{reg } I(G) = \nu(G) + 1$ and $\text{reg } I(G) = \nu(G) + 2$.

Lemma 3.18. *Adopt [Notation 3.13](#). If $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) = 1$ then*

$$\text{reg } I(G) = \nu(G) + 2.$$

Proof. From [Theorem 3.17](#) we have that $\text{reg } (I(G)) \leq \nu(G) + 2$. Using the same method as in [Theorem 3.17](#), we can obtain a lower bound

$$\text{reg } I(G) \geq \text{reg } I(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(G \setminus \Gamma_G(C_n \cup C_m)) + 3 = \nu(G) + 2,$$

and so the equality follows. \square

Lemma 3.19. *Adopt [Notation 3.13](#). If $\nu(G) = \nu(G \setminus \Gamma_G(C_n))$ then*

$$\text{reg } I(G) \geq \nu(G) + 2.$$

Symmetrically, the same argument holds for C_m .

Proof. The proof follows similarly to [Theorem 3.12](#). From [Remark 3.14\(i\)](#), [Corollary 1.14](#), [Theorem 1.20](#) and [Theorem 1.16](#) we get

$$\begin{aligned} \text{reg } I(G \setminus \Gamma_G(C_n)) &= \text{reg } I(C_n) + \text{reg } I(\cup_{i=1}^c H_i) - 1 \\ &\geq (\nu(C_n) + 2) + (\nu(\cup_{i=1}^c H_i) + 1) - 1 \\ &\geq \nu(G \setminus \Gamma_G(C_n)) + 2 \\ &\geq \nu(G) + 2. \end{aligned}$$

So the inequality follows from the fact that $G \setminus \Gamma_G(C_n)$ is an induced subgraph of G . \square

The following simple logical argument will be used several times in the next theorem.

Observation 3.20. Let P_1, P_2, P_3 be boolean values, (i.e. true or false). Assume that P_1 is true if and only if P_2 and P_3 are true, that is

$$P_1 \iff (P_2 \wedge P_3).$$

Suppose that if P_2 is true then P_3 is false, that is

$$P_2 \implies \neg P_3.$$

Then, P_1 is false.

Notation 3.21. Let X be a mathematical expression. Then, $P[X]$ represents a boolean value, which is true if X is satisfied and false otherwise.

Taking into account the induced matching numbers $\nu(G)$, $\nu(G \setminus \Gamma_G(C_n \cup C_m))$, $\nu(G \setminus \Gamma_G(C_n))$ and $\nu(G \setminus \Gamma_G(C_m))$, we can give necessary and sufficient conditions for the equality $\text{reg } I(G) = \nu(G) + 1$.

Theorem 3.22. Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod{3}$ and $l \geq 3$. Then $\text{reg } I(G) = \nu(G) + 1$ if and only if the following conditions are satisfied:

- (i) $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$;
- (ii) $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$;
- (iii) $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$.

Proof. From [Theorem 3.17](#), [Lemma 3.18](#) and [Lemma 3.19](#), we have that the conditions (i), (ii) and (iii) are necessary. Hence, it is enough to prove $\text{reg } I(G) \leq \nu(G) + 1$ under these conditions.

For any $x \in G$ we denote $G' = G \setminus x$ and $G'' = G \setminus N[x]$. Then, we have the upper bound

$$\text{reg } I(G) \leq \max\{\text{reg } I(G'), \text{reg } I(G'') + 1\}.$$

We shall prove that under the conditions (i), (ii) and (iii) there exists a vertex $x \in C_n$ such that $\text{reg } I(G') \leq \nu(G) + 1$ and $\text{reg } I(G'') + 1 \leq \nu(G) + 1$. We divide the proof into three steps.

Step 1. In this step we prove that for any $x \in C_n$ we have $\text{reg } I(G') \leq \nu(G) + 1$. First we note the following two observations:

- It follows from [Theorem 1.18](#) that $\text{reg } I(G') \leq \nu(G') + 2$. Hence, $\nu(G') < \nu(G)$ implies that $\text{reg } I(G') \leq \nu(G') + 2 \leq \nu(G) + 1$.
- Since G' is a unicyclic graph, [\[2, Theorem 1.2\]](#) implies that $\text{reg } I(G') = \nu(G') + 2$ if and only if $\nu(G') = \nu(G' \setminus \Gamma_{G'}(C_m))$.

Thus, it follows that

$$\text{reg } I(G') = \nu(G) + 2 \iff \left(\nu(G) = \nu(G') \text{ and } \nu(G') = \nu(G' \setminus \Gamma_{G'}(C_m)) \right).$$

In [Observation 3.20](#), let $P_1 = P[\text{reg } I(G') = \nu(G) + 2]$, $P_2 = P[\nu(G) = \nu(G')]$ and $P_3 = P[\nu(G') = \nu(G' \setminus \Gamma_{G'}(C_m))]$. From the logical argument of [Observation 3.20](#), if we prove that $\nu(G') = \nu(G)$ implies $\nu(G') > \nu(G' \setminus \Gamma_{G'}(C_m))$ then we get the desired inequality $\text{reg } I(G') \leq \nu(G) + 1$. Assume that $\nu(G) = \nu(G')$. From the hypothesis $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$ and the fact that $G' \setminus \Gamma_{G'}(C_m)$ is an induced subgraph of $G \setminus \Gamma_G(C_m)$, then we get

$$\nu(G') = \nu(G) > \nu(G \setminus \Gamma_G(C_m)) \geq \nu(G' \setminus \Gamma_{G'}(C_m)).$$

Therefore, we have $\text{reg } I(G') \leq \nu(G) + 1$.

Step 2. Since $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$, it follows from [Remark 3.14\(iv\)](#) that there exists some $1 \leq i \leq c$ such that $\nu(F_i) > \nu(H_i)$. Following [Notation 3.13](#), we have that F_1 is a

unicyclic graph containing the cycle C_m and that F_i is a tree for all $i > 1$. In this step, fix $i > 1$ where F_i is a tree and $\nu(F_i) > \nu(H_i)$.

Let x be the vertex in $F_i \cap C_n$ and H be the induced subgraph $H = G \setminus (F_i \cup N[x])$. Note that $G'' = H \cup H_i$, $d(H, H_i) \geq 2$ and $d(H, F_i) \geq 2$. Then

$$\nu(G'') = \nu(H) + \nu(H_i) < \nu(H) + \nu(F_i) \leq \nu(G)$$

follows from the condition $\nu(H_i) < \nu(F_i)$. So we have that $\nu(G'') < \nu(G)$.

As in Step 1, we note the following two observations:

- It follows from [Theorem 1.18](#) that $\text{reg } I(G'') \leq \nu(G'') + 2$. Hence, $\nu(G'') + 1 < \nu(G)$ implies that $\text{reg } I(G'') + 1 \leq \nu(G'') + 3 \leq \nu(G) + 1$.
- Since G'' is a unicyclic graph, [\[2, Theorem 1.2\]](#) implies that $\text{reg } I(G'') = \nu(G'') + 2$ if and only if $\nu(G'') = \nu(G'' \setminus \Gamma_{G''}(C_m))$.

So, we have that

$$\text{reg } I(G'') + 1 = \nu(G) + 2 \iff \left(\nu(G) = \nu(G'') + 1 \text{ and } \nu(G'') = \nu(G'' \setminus \Gamma_{G''}(C_m)) \right).$$

Let K be the induced subgraph defined by $K = (G \setminus \Gamma_G(C_m)) \setminus (F_i \cup N[x])$. Since $i > 1$ then $F_i \cap F_1 = \emptyset$, and so we get the following statements:

- $G'' \setminus \Gamma_{G''}(C_m) = K \cup H_i$, because $G'' = H \cup H_i$ where $H = G \setminus (F_i \cup N[x])$, $C_m \subset H$ and $d(C_m, H_i) \geq 2$.
- $K \cup F_i$ is an induced subgraph of $G \setminus \Gamma_G(C_m)$.
- Since $d(K, F_i) \geq 2$ and $d(K, H_i) \geq 2$, we have the following inequalities

$$\nu(G'' \setminus \Gamma_{G''}(C_m)) = \nu(K) + \nu(H_i) < \nu(K) + \nu(F_i) \leq \nu(G \setminus \Gamma_G(C_m)).$$

In [Observation 3.20](#), let $P_1 = P[\text{reg } I(G'') + 1 = \nu(G) + 2]$, $P_2 = P[\nu(G) = \nu(G'') + 1]$ and $P_3 = P[\nu(G'') = \nu(G'' \setminus \Gamma_{G''}(C_m))]$. So it is enough to prove that $\nu(G) = \nu(G'') + 1$ implies $\nu(G'') > \nu(G'' \setminus \Gamma_{G''}(C_m))$. Assuming $\nu(G) = \nu(G'') + 1$ then we get

$$\nu(G'') = \nu(G) - 1 > \nu(G \setminus \Gamma_G(C_m)) - 1 \geq \nu(G'' \setminus \Gamma_{G''}(C_m)).$$

Therefore, in this case we have $\text{reg } I(G'') + 1 \leq \nu(G) + 1$.

Step 3. In this last step we assume that $\nu(F_1) > \nu(H_1)$ and that $\nu(F_i) = \nu(H_i)$ for all $i > 1$. Let x be the vertex in $F_1 \cap C_n$, then as in Step 2 we have that:

- $\nu(G'') < \nu(G)$.
- $\text{reg } I(G'') + 1 = \nu(G) + 2 \iff \left(\nu(G) = \nu(G'') + 1 \text{ and } \nu(G'') = \nu(G'' \setminus \Gamma_{G''}(C_m)) \right)$.

Once more, if we prove that $\nu(G) = \nu(G'') + 1$ implies $\nu(G'') > \nu(G'' \setminus \Gamma_{G''}(C_m))$ then we obtain that $\text{reg } I(G'') + 1 \leq \nu(G) + 1$.

We denote by L the induced subgraph of $G'' \setminus \Gamma_{G''}(C_m)$ given by

$$L = (G'' \setminus \Gamma_{G''}(C_m)) \setminus \Gamma_G(C_n).$$

Due to [Remark 3.14\(ii\)](#), the graph L has the decomposition

$$L = (C_n \setminus N_{C_n}[x]) \cup \left(\bigcup_{i=1}^c H_i \right) \cup (H_1 \setminus \Gamma_{H_1}(C_m)),$$

with all the disjoint components at distance at least two between each other, and so we have

$$\nu(L) = \nu(C_n \setminus N_{C_n}[x]) + \sum_{i=2}^c \nu(H_i) + \nu(H_1 \setminus \Gamma_{H_1}(C_m)).$$

By proceeding as in the proofs of [Lemma 3.10](#) or [Lemma 3.15](#), from the conditions $\nu(F_i) = \nu(H_i)$ for all $i > 1$, we obtain

$$\begin{aligned} \nu(L) &= \nu((C_n \setminus N_{C_n}[x])) + \sum_{i=2}^c \nu(F_i) + \nu(H_1 \setminus \Gamma_{H_1}(C_m)) \\ &\geq \nu(G'' \setminus \Gamma_{G''}(C_m)). \end{aligned}$$

Thus, $\nu(L) = \nu(G'' \setminus \Gamma_{G''}(C_m))$ because L is an induced subgraph of $G'' \setminus \Gamma_{G''}(C_m)$. We also have that L is an induced subgraph of $G \setminus \Gamma_G(C_n \cup C_m)$ because we have the equality

$$L = (G \setminus \Gamma_G(C_n \cup C_m)) \setminus N[x].$$

Finally, from the hypothesis $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$ we can obtain

$$\nu(G'') = \nu(G) - 1 > \nu(G \setminus \Gamma_G(C_n \cup C_m)) \geq \nu(L) = \nu(G'' \setminus \Gamma_{G''}(C_m)).$$

Therefore, in this case we also have $\text{reg } I(G'') + 1 \leq \nu(G) + 1$.

So, we have finished the proof. \square

3.4. Case IV. In this short subsection we deal with the remaining case, we assume that G is a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod{3}$ and $l \leq 2$.

When $l \leq 2$, the two cycles are too close to each other, and it is difficult to make a direct analysis (with our methods). Fortunately, using the complete characterization of the case $l \geq 3$, the problem can be solved with the Lozin transformation. Suppose that x is a vertex on the bridge P_l (at most two), then we apply the Lozin transformation of G with respect to x , and obtain a bicyclic graph $\mathcal{L}_x(G)$ with dumbbell of the type $C_n \cdot P_k \cdot C_m$ where $k \geq 4$. From [\[25, Lemma 1\]](#) and [\[7, Theorem 1.1\]](#) we get the equality

$$(4) \quad \text{reg}(I(\mathcal{L}_x(G))) - \nu(\mathcal{L}_x(G)) = \text{reg}(I(G)) - \nu(G).$$

Therefore we obtain a characterization in the following corollary.

Corollary 3.23. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod{3}$ and $l \leq 2$. Let x be a point on the bridge P_l and let $\mathcal{L}_x(G)$ be the Lozin transformation of G with respect to x . Then we have that $\nu(G) + 1 \leq \text{reg } I(G) \leq \nu(G) + 2$, and that $\text{reg } I(G) = \nu(G) + 1$ if and only if the following conditions are satisfied:*

- (i) $\nu(\mathcal{L}_x(G)) - \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n \cup C_m)) > 1$;
- (ii) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n))$;
- (iii) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_m))$.

Proof. From [Proposition 3.3](#), it follows that $\nu(G) + 1 \leq \text{reg } I(G) \leq \nu(G) + 2$. Due to (4), we can apply the Lozin transformation and reduce the problem to the case where the bridge has more than three vertices. Finally, [Theorem 3.22](#) gives us the result. \square

3.5. The characterization. Finally, the theorem below contains the characterization that we found.

Theorem 3.24. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. The following statements hold.*

- (I) *Let $n, m \equiv 0, 1 \pmod{3}$, then $\text{reg } I(G) = \nu(G) + 1$.*
- (II) *Let $n \equiv 0, 1 \pmod{3}$ and $m \equiv 2 \pmod{3}$, then*

$$\nu(G) + 1 \leq \text{reg } I(G) \leq \nu(G) + 2,$$

and $\text{reg } I(G) = \nu(G) + 2$ if and only if $\nu(G) = \nu(G \setminus \Gamma_G(C_m))$.

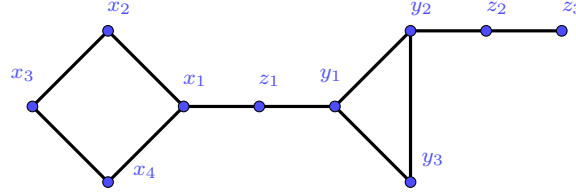
- (III) *Let $n, m \equiv 2 \pmod{3}$ and $l \geq 3$, then $\nu(G) + 1 \leq \text{reg } I(G) \leq \nu(G) + 3$. Moreover:*
 - (i) *$\text{reg } I(G) = \nu(G) + 3$ if and only if $\nu(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(G)$.*

- (ii) $\text{reg } I(G) = \nu(G) + 1$ if and only if the following conditions hold:
- (a) $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$;
 - (b) $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$;
 - (c) $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$.
- (IV) Let $n, m \equiv 2 \pmod{3}$ and $l \leq 2$, then $\nu(G) + 1 \leq \text{reg } I(G) \leq \nu(G) + 2$. If x is an edge on P_l and $\mathcal{L}_x(G)$ be the Lozin transformation of G with respect to x , then $\text{reg } I(G) = \nu(G) + 1$ if and only if the following conditions are satisfied:
- (a) $\nu(\mathcal{L}_x(G)) - \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n \cup C_m)) > 1$;
 - (b) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n))$;
 - (c) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_m))$.

Proof. Statement (I) follows from Proposition 3.3. In Theorem 3.12, (II) is proved. By Theorem 3.17 and Theorem 3.22, we get (III). Finally, from Corollary 3.23, we obtain (IV). \square

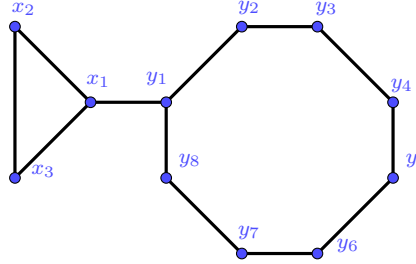
3.6. Examples. In this last subsection we give examples for each one of the statements in the characterization of Theorem 3.24.

Example 3.25. Statement (I) of Theorem 3.24. Let G be the graph below.



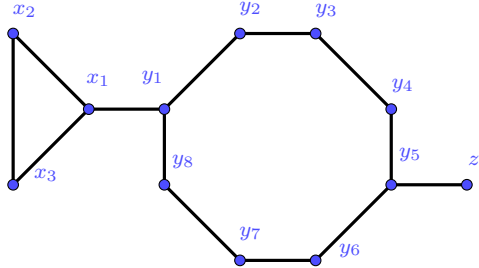
Then we have $\text{reg } I(G) = 4$ and $\nu(G) = 3$.

Example 3.26. Statement (II) of Theorem 3.24. Let G be the graph below.



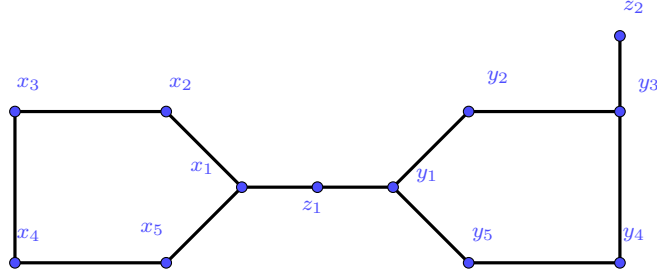
Then we have $\text{reg } I(G) = 5$ and $\nu(G) = 3$.

If G is the graph given below, then we have $\text{reg } I(G) = 5$ and $\nu(G) = 4$.



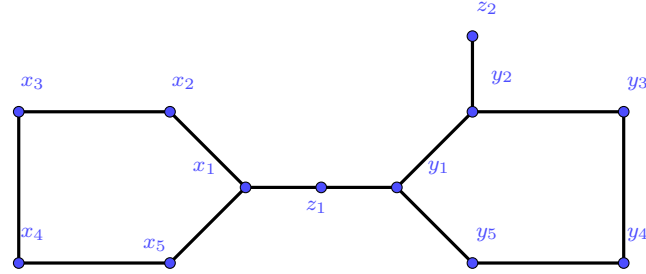
Example 3.27. Statement (III) of Theorem 3.24.

Let G be the graph given in [Example 3.1](#). Then we have $\text{reg } I(G) = 6$ and $\nu(G) = 3$.
 Let G be the graph below.



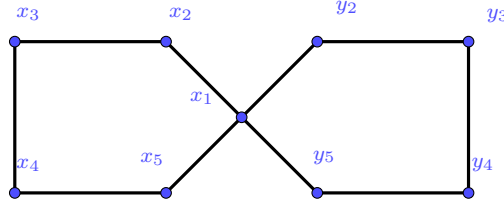
Then we have $\text{reg } I(G) = 5$ and $\nu(G) = 3$.

Let G be the graph given below and obtained by moving the whisker to the left.



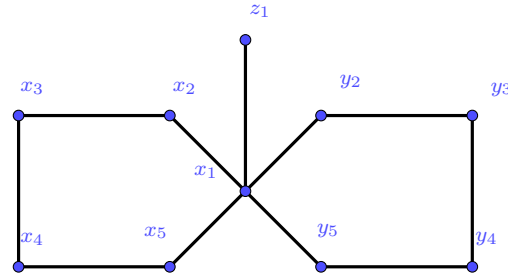
Then we have $\text{reg } I(G) = 5$ and $\nu(G) = 4$.

Example 3.28. Statement (IV) of [Theorem 3.24](#). Let G be the graph below.



Then we have $\text{reg } I(G) = 4$ and $\nu(G) = 2$.

Let G be the graph given below and obtained by adding a whisker to the above graph at the join vertex x_1 .



Then we have $\text{reg } I(G) = 4$ and $\nu(G) = 3$.

4. CASTELNUOVO-MUMFORD REGULARITY OF POWERS

In this section, we study the regularity of the powers of $I(C_n \cdot P_l \cdot C_m)$ when $l \leq 2$. Our strategy to compute $\text{reg } I(C_n \cdot P_l \cdot C_m)^q$ for $q \geq 1$ relies on finding an upper bound and a lower bound on $\text{reg } I(C_n \cdot P_l \cdot C_m)^q$ where these bounds coincide and are equal to

$$2q + \text{reg } I(C_n \cdot P_l \cdot C_m) - 2.$$

In order to obtain an upper bound, we follow the even-connection argument given in [4, Theorem 5.2]. Then we proceed by looking at “nice” induced subgraphs of $C_n \cdot P_l \cdot C_m$ and we find a lower bound on $\text{reg } I(C_n \cdot P_l \cdot C_m)^q$ which is equal to the found upper bound.

Let I be an arbitrary ideal generated in degree d and let $b_q := \text{reg } (I^q) - dq$ for $q \geq 1$. An interesting question is to study of the sequence $\{b_i\}_{i \geq 1}$. In [12] Eisenbud and Harris proved that if $\dim(R/I) = 0$, then $\{b_i\}_{i \geq 1}$ is a weakly decreasing sequence of non-negative integers. In [5] Banerjee, Beyarslan and Hà conjectured that for any edge ideal, $\{b_i\}_{i \geq 1}$ is a weakly decreasing sequence (see [5, Conjecture 7.11]). For the edge ideal of any dumbbell graph with $l \leq 2$, we prove $b_i = b_1$ for all $i \geq 1$. However, we expect $b_i \leq b_1$ for all $i \geq 1$ for any graph.

Remark 4.1. From Theorem 2.4 and Theorem 2.16, for any $l \leq 2$ we have that

$$\text{reg } I(C_n \cdot P_l \cdot C_m) \geq \left\lfloor \frac{n + m + l + 1}{3} \right\rfloor.$$

The previous inequality is not satisfied when $l \geq 3$, because $\text{reg } I(C_4 \cdot P_3 \cdot C_4) = 3$ and $\left\lfloor \frac{4+4+3+1}{3} \right\rfloor = 4$.

As recalled earlier, we use the notation of even-connection from Banerjee [4, Theorem 5.2]. The following lemma is important in our treatment of the even-connected vertices, and its proof is similar to [4, Lemma 6.13].

Lemma 4.2. *Let G be a graph. As in Remark 1.10, let G' be the graph associated to $(I(G)^{q+1} : e_1 \cdots e_q)^{\text{pol}}$. Suppose $u = p_0, p_1, \dots, p_{2s+1} = v$ is a path that even-connects u and v with respect to the q -fold $e_1 \cdots e_q$. Then we have*

$$\bigcup_{i=0}^{2s+1} N_{G'}[p_i] \subset N_{G'}[u] \cup N_{G'}[v].$$

Proof. Let U be the set of vertices $U = \{p_0, p_1, \dots, p_{2s+1}\}$. For each $1 \leq k \leq s$ we have that $p_{2k-1}p_{2k} = e_{j_k}$ for some $1 \leq j_k \leq q$, i.e. u and v are even connected with respect to the s -fold $e_{j_1}e_{j_2} \cdots e_{j_s}$.

Let w be a vertex even-connected to some vertex $z \in U$ with respect to the q -fold $e_1 \cdots e_q$. Then, there exists a path $z = r_0, r_1, \dots, r_{2t+1} = w$ that even-connects z and w with respect to the q -fold $e_1 \cdots e_q$. Let i be the largest integer such that $r_i \in U$. From the fact that $r_0 = z \in U$, we have that the integer i is well defined and $i \geq 0$. Let k be an integer such that $p_k = r_i$.

The proof is now divided into four different cases depending on $i \bmod 2$ and $k \bmod 2$. When i and k are both odd integers, we have that $r_i r_{i+1}$ is equal to some edge of $\{e_1, e_2, \dots, e_q\}$ and that $p_{k-1}p_k$ is not equal to any edge of $\{e_{j_1}, e_{j_2}, \dots, e_{j_s}\}$. By the definition of i we have

$$\{r_{i+1}, r_{i+2}, \dots, r_{2t+1}\} \cap U = \emptyset.$$

So, in this case, it follows that

$$u = p_0, \dots, p_{k-1}, p_k = r_i, r_{i+1}, \dots, r_{2t+1} = w$$

is a path that even-connects u and w with respect to the q -fold $e_1 \cdots e_q$.

The other three cases follow in a similar way. Therefore, we have that if w even-connected to some $z \in U$, then w is even-connected to either u or v .

Now, we only need to prove that any $w \in N_G[z]$ for some $z \in U$ is even-connected to either u or v . This part is simple, if $z = p_{2j}$ then $u = p_0, \dots, p_{2j} = z, w$ is a path that even-connects u and w , otherwise, if $z = p_{2j-1}$ then $w, z = p_{2j-1}, \dots, p_{2s+1} = v$ is a path that even-connects w and v .

So we are done. \square

The next lemma is similar to [6, Lemma 5.1], but adapted to the current setting of a dumbbell.

Lemma 4.3. *Let $G = C_n \cdot P_l \cdot C_m$. If $(I(G)^{q+1} : e_1 \cdots e_q)$ is not a square-free monomial ideal and G' is the associated graph, then there exists a vertex z which is even-connected to itself. Then, G' has a leaf and $N_{G'}[z]$ contains one of the two cycles. In particular, if we denote the corresponding leaf by e , then G'_e is an induced subgraph of a unicyclic graph.*

Proof. Suppose $z = p_0, p_1, \dots, p_{2l+1} = z$ is an even-connection of z with itself. Let $0 \leq a < b \leq 2l + 1$ be integers such that $p_a, p_{a+1}, \dots, p_b = p_a$ is an even-connection and $b - a$ is minimal. Then, $p_a, p_{a+1}, \dots, p_b = p_a$ is a simple closed path lying on $C_n \cdot P_l \cdot C_m$ and so it is necessarily equal to either C_n or C_m .

Finally, Lemma 4.2 implies that $N_{G'}[z]$ contains either C_n or C_m . \square

Lemma 4.4. *Let $G = C_n \cdot P_l \cdot C_m$ with $l \leq 2$ and H be a graph such that G is a subgraph of H with the same set of vertices (i.e., $V(H) = V(G)$ and $E(H) \supseteq E(G)$). For any two vertices $u, v \in H$ such that $\{u, v\} \notin E(G)$, we have that*

$$\text{reg } I(H \setminus (N_H[u] \cup N_H[v])) \leq \text{reg } I(G) - 1.$$

Proof. Let $K = N_G[u] \cap N_G[v]$. We divide the proof according to the cardinality $|K|$ of K . Notice that for the dumbbell G we always have $0 \leq |K| \leq 2$.

Since $H \setminus (N_H[u] \cup N_H[v])$ is an induced subgraph of $H \setminus (N_G[u] \cup N_G[v])$, from Theorem 1.7(i), it is enough to prove that $\text{reg } I(H \setminus (N_G[u] \cup N_G[v])) \leq \text{reg } I(G) - 1$.

Step 1. Suppose that $|K| = 0$. Then, the graph $H \setminus (N_G[u] \cup N_G[v])$ is obtained by deleting at least 6 vertices, and so $|H \setminus (N_G[u] \cup N_G[v])| \leq |G| - 6 \leq n + m + l - 8$. Note that we can add two vertices to $H \setminus (N_G[u] \cup N_G[v])$ and connect them in such a way that we obtain a Hamiltonian path. Let L be a graph obtained by adding two vertices and certain edges connecting these two new vertices, such that L has a Hamiltonian path. Since $|L| \leq n + m + l - 6$, Theorem 1.22 yields

$$\text{reg } I(L) \leq \left\lfloor \frac{n + m + l - 5}{3} \right\rfloor + 1 = \left\lfloor \frac{n + m + l + 1}{3} \right\rfloor - 1,$$

and by applying Remark 4.1, we get $\text{reg } I(L) \leq \text{reg } I(G) - 1$. Since $H \setminus (N_G[u] \cup N_G[v])$ is an induced subgraph of L , Theorem 1.7(i) implies that $\text{reg } I(H \setminus (N_G[u] \cup N_G[v])) \leq \text{reg } I(G) - 1$.

Step 2. Suppose that $|K| = 1$. Here the proof follows along the same lines of Step 1. In this case the graph $H \setminus (N_G[u] \cup N_G[v])$ is obtained by deleting at least 5 vertices. Now, note that we can add one vertex to $H \setminus (N_G[u] \cup N_G[v])$ and connect it in such a way that we obtain a Hamiltonian path. Let L be a graph obtained by adding one vertex and certain edges connecting this new vertex, such that L has a Hamiltonian path. Since $|L| \leq (n + m + l - 2) - 5 + 1 = n + m + l - 6$, then the rest of the proof follows as in Step 1.

Step 3. Suppose that $|K| = 2$. In this case, note that one of the cycles is necessarily equal to C_4 , say $C_n = C_4$, and that $u, v \in C_4$ with $\{u, v\} \notin E(G)$. Hence, it follows that $H \setminus (N_G[u] \cup N_G[v])$ has a Hamiltonian path with $\leq m$ vertices if $l = 2$ and $\leq m - 1$ vertices

if $l = 1$. From [Theorem 1.22](#) and [Remark 4.1](#), then we have $\text{reg } I(H \setminus (N_G[u] \cup N_G[v])) \leq \text{reg } I(G) - 1$.

So we are through. \square

Theorem 4.5. *Let $G = C_n \cdot P_l \cdot C_m$ with $l \leq 2$ and $I = I(G)$ be its edge ideal, then*

$$\text{reg } (I^{q+1} : e_1 \cdots e_q) \leq \text{reg } I$$

for any $1 \leq q$ and any edges $e_1, \dots, e_q \in E(G)$.

Proof. We split the proof into two cases.

Case 1. First, suppose $(I^{q+1} : e_1 \cdots e_q)$ is a square-free monomial ideal. In this case $(I^{q+1} : e_1 \cdots e_q) = I(G')$ where G' is a graph with $V(G) = V(G')$ and $E(G) \subseteq E(G')$. Let $E(G') = E(G) \cup \{a_1, \dots, a_r\}$, then each edge a_i is induced from even-connecting two different vertices (i.e., each a_i is not a whisker). By [Theorem 1.7](#), we have

$$\text{reg } I(G') \leq \max\{\text{reg } I(G' \setminus a_1), \text{reg } I(G'_{a_1}) + 1\}$$

Since $a_1 \notin E(G)$, [Lemma 4.4](#) implies that $\text{reg } I(G'_{a_1}) + 1 \leq \text{reg } I(G)$.

In the same way, for any subgraph $H = G' \setminus \{a_1, \dots, a_i\}$, since $V(H) = V(G)$ and $E(H) \supseteq E(G)$, [Lemma 4.4](#) also gives us that

$$\text{reg } (I(H_{a_{i+1}})) + 1 \leq \text{reg } (I(G)).$$

By continuing this process, we get $\text{reg } I(G') \leq \text{reg } I(G)$.

Case 2. Suppose $(I^{q+1} : e_1 \cdots e_q)$ is not square-free and G' is the graph associated to $(I^{q+1} : e_1 \cdots e_q)^{\text{pol}}$. Let $\{b_1, b_2, \dots, b_s\}$ be the subset of edges of $E(G') \setminus E(G)$ that are generated by square monomials (i.e., each b_i is a whisker).

From [Theorem 1.7](#) we have the inequality

$$\text{reg } I(G') \leq \max\{\text{reg } I(G' \setminus b_1), 1 + \text{reg } I(G'_{b_1})\}.$$

[Lemma 4.3](#) implies that one of the cycles is deleted from G'_{b_1} , then there exists an edge $e \in G$ such that $d(e, G'_{b_1}) \geq 2$. So, for such an edge e we get that the disjoint union $G'_{b_1} \cup e$ is an induced subgraph of $G' \setminus b_1$. Thus, [Theorem 1.7](#) and [Corollary 1.14](#) yield that

$$\text{reg } (I(G'_{b_1})) + 1 = \text{reg } (I(G'_{b_1} \cup e)) \leq \text{reg } (I(G' \setminus b_1)).$$

Therefore, we obtain that $\text{reg } I(G') \leq \text{reg } I(G' \setminus b_1)$.

By applying the same argument, it follows that

$$\text{reg } I(G') \leq \text{reg } I(G' \setminus b_1) \leq \text{reg } I(G' \setminus \{b_1, b_2\}) \leq \cdots \leq \text{reg } I(G' \setminus \{b_1, \dots, b_s\}).$$

Since the graph $G' \setminus \{b_1, \dots, b_s\}$ has no whiskers, then [Step 1](#) implies that

$$\text{reg } I(G') \leq \text{reg } I(G' \setminus \{b_1, \dots, b_s\}) \leq \text{reg } I(G).$$

Therefore, the proof is completed. \square

Remark 4.6. *The previous theorem is a generalization of a work done by Gu in [\[15\]](#) for the case $l = 1$.*

Theorem 4.7. *For the dumbbell graph $C_n \cdot P_l \cdot C_m$ with $l \leq 2$, we have*

$$\text{reg } (C_n \cdot P_l \cdot C_m)^q \geq 2q + \text{reg } I(C_n \cdot P_l \cdot C_m) - 2,$$

for any $q \geq 1$.

Proof. Using the inequality $\text{reg } I(C_n \cdot P_2 \cdot C_m)^q \geq 2q + \nu(C_n \cdot P_2 \cdot C_m) - 1$ of [Theorem 1.21](#), for the cases where $\text{reg } I(C_n \cdot P_l \cdot C_m) = \nu(C_n \cdot P_l \cdot C_m) + 1$ we get the expected inequality. We divide the proof in two halves, the cases $l = 1$ and $l = 2$.

Case 1. Let $l = 1$. We only need to focus on the case where $n, m \equiv 2 \pmod{3}$. Let H be the induced subgraph of $C_n \cdot P_1 \cdot C_m$ mentioned in the proof of [Theorem 2.7](#), i.e. $H = (C_n \cdot P_1 \cdot C_m) \setminus \{x_n\} = P_{n-1} \cdot C_m$. Using [Theorem 2.4](#), [Proposition 2.3](#) and the modularity $n, m \equiv 2 \pmod{3}$, we can check that

$$\nu(H) = \nu(C_n \cdot P_1 \cdot C_m)$$

and that

$$\nu(H) = \nu(H \setminus \Gamma_H(C_m)).$$

From [Theorem 2.7](#) and [\[2, Theorem 1.2\]](#) we get

$$\text{reg } I(C_n \cdot P_1 \cdot C_m) = \nu(C_n \cdot P_1 \cdot C_m) + 2 = \nu(H) + 2 = \text{reg } I(H).$$

Since H is an induced subgraph of $C_n \cdot P_1 \cdot C_m$, then from [\[2, Theorem 1.1\]](#) and [\[6, Corollary 4.3\]](#) we get the inequality

$$\text{reg } I(C_n \cdot P_1 \cdot C_m)^q \geq \text{reg } I(H)^q = 2q + \text{reg } I(H) - 2 = 2q + \text{reg } I(C_n \cdot P_1 \cdot C_m) - 2.$$

Case 2. Let $l = 2$. We only need to focus on the cases where $n \equiv 0, 1 \pmod{3}$ and $m \equiv 2 \pmod{3}$. We take the same induced subgraph H as in [Lemma 2.12](#). The induced subgraph $H = (C_n \cdot P_2 \cdot C_m) \setminus \{x_1\}$ of $C_n \cdot P_2 \cdot C_m$ is given as the union of a path of length $n - 1$ and the cycle C_m , i.e., $H = P_{n-1} \cup C_m$.

By [Theorem 2.13](#), for the cases $n \equiv 0, 1 \pmod{3}$ and $m \equiv 2 \pmod{3}$, we have

$$\text{reg } I(C_n \cdot P_2 \cdot C_m) = \nu(C_n \cdot P_2 \cdot C_m) + 2 = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2,$$

and from [Corollary 1.14](#), [Theorem 1.15](#) and [Theorem 1.20](#) we obtain

$$\text{reg } I(H) = \text{reg } (I(P_{n-1})) + \text{reg } (I(C_m)) - 1 = \nu(P_{n-1}) + \nu(C_m) + 2 = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2.$$

Hence, we get $\text{reg } I(C_n \cdot P_2 \cdot C_m) = \text{reg } I(H)$. Finally, using [\[2, Theorem 1.1\]](#) and [\[6, Corollary 4.3\]](#), we get the inequality

$$\text{reg } I(C_n \cdot P_2 \cdot C_m)^q \geq \text{reg } I(H)^q = 2q + \text{reg } I(H) - 2 = 2q + \text{reg } I(C_n \cdot P_2 \cdot C_m) - 2.$$

Therefore, the proof is completed. \square

Theorem 4.8. *For the dumbbell graph $C_n \cdot P_l \cdot C_m$ with $l \leq 2$, we have*

$$\text{reg } I(C_n \cdot P_l \cdot C_m)^q = 2q + \text{reg } I(C_n \cdot P_l \cdot C_m) - 2$$

for all $q \geq 1$.

Proof. It follows by [Theorem 4.5](#), [Theorem 1.11](#) and [Theorem 4.7](#). \square

Remark 4.9. *One may ask whether*

$$\text{reg } I(C_n \cdot P_l \cdot C_m)^q = 2q + \text{reg } I(C_n \cdot P_l \cdot C_m) - 2$$

always holds for given n, m, l and q . Unfortunately, it is no longer true for any n, m, l and q as it can be seen from the following example:

$$6 = \text{reg } I(C_5 \cdot P_3 \cdot C_5)^2 < 4 + \text{reg } I(C_5 \cdot P_3 \cdot C_5) - 2 = 7.$$

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